



# Some approximation and regularity results for fully nonlinear elliptic and parabolic equations

Jean-Paul Daniel

## ► To cite this version:

Jean-Paul Daniel. Some approximation and regularity results for fully nonlinear elliptic and parabolic equations. General Mathematics [math.GM]. Université Pierre et Marie Curie - Paris VI, 2014. English. NNT : 2014PA066468 . tel-01165018

**HAL Id: tel-01165018**

**<https://theses.hal.science/tel-01165018>**

Submitted on 18 Jun 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Quelques résultats d'approximation et de régularité pour des équations elliptiques et paraboliques non-linéaires

## THÈSE

présentée et soutenue publiquement le 12 décembre 2014

pour l'obtention du grade de

Docteur de l'Université Pierre et Marie Curie

Spécialité Mathématiques Appliquées

par

Jean-Paul DANIEL

### Composition du jury

<i>Rapporteurs :</i>	Francesca DA LIO Olivier LEY
<i>Examineurs :</i>	Yves ACHDOU Pierre CARDALIAGUET Cyril IMBERT
<i>Directeurs de thèse :</i>	Scott N. ARMSTRONG Sylvia SERFATY



## Remerciements

Je tiens tout d'abord à remercier mes directeurs de thèse, Scott N. Armstrong et Sylvia Serfaty, de m'avoir proposé un sujet d'étude passionnant et de m'avoir encadré. Ils ont su me communiquer leur goût pour les mathématiques en me faisant profiter de leur expérience, de leur vision de la recherche. J'ai beaucoup apprécié la liberté qu'ils m'ont laissée dans le choix des sujets à aborder, tout en étant toujours disponibles pour répondre à mes questions et m'apporter leur aide précieuse. Leurs points de vue et leurs conseils m'ont été très utiles et m'ont permis de mener à bien mes travaux de recherche. Ce sont des mathématiciens extrêmement brillants mais également des travailleurs acharnés, qui resteront tous les deux pour moi des modèles et des sources d'inspiration.

Olivier Ley et Francesca Da Lio m'ont fait l'honneur d'accepter le rôle difficile de rapporteurs, je les remercie vivement. Je suis très reconnaissant à Pierre Cardaliaguet, Cyril Imbert et Yves Achdou d'avoir accepté de faire partie de mon jury de thèse.

Je remercie Isabelle Gallagher de m'avoir encadré pendant le stage de Master. Je remercie également Guy Barles pour une discussion particulièrement intéressante sur le problème de Neumann lors d'une école d'été sur les équations HJB à Cetraro.

Je tiens à remercier les directeurs du Laboratoire Jacques-Louis Lions, Yvon Maday puis Benoît Perthame, et tous les membres du laboratoire, de m'avoir accueilli et de m'avoir donné des conditions exceptionnelles pour réaliser ce travail. Je remercie en particulier Marie Postel et Amaury Lambert (LPMA) pour qui j'ai effectué des TD et Frédéric Coquel pour leur gentillesse naturelle. Je remercie Salima, Catherine et Nadine ainsi que Danielle Boulic et Liliane Ruprecht pour leur aide, leur efficacité et leur gentillesse. Merci également à toute l'équipe informatique Khashayar, Hugues et Altaïr pour avoir toujours été extrêmement prompts à résoudre tous les questions qui ont pu se produire et pour leur sympathie.

Je tiens à remercier mes collègues doctorants, Pierre, Nastasia, Maxime, Sarah, Jan, Thibaut L., Benjamin, Magali, Waafa, Thibault B., Ange, Thomas, Dena, Mamadou, Charles (toujours disponible pour une discussion tennistique, suivre un inoubliable Chardy-Raonic, discuter de la régularité de la fonction distance ou une bière), Nicole, Lise-Marie, Oana, Abdel, Khaled, Kamel, Philippe, Mehdi, Nico, Thomas L., ou certains anciens comme Marie K. ou Dorian. Je remercie les anciens doctorants aujourd'hui maîtres de conférences ou chargés de recherche : Nicolas R., Pierre G., Sepideh, Vincent, Alexander et Laurent M..

Je tiens à remercier l'ensemble du bureau 15-16-221 : Ludovic, Sylvain (et Séverine), Juliette, Malik, Casimir, Giacomo, la machine à café et Guillaume, les nouveaux Shuyian et Carlos, ainsi que Laurent. Je remercie André pour tout ce qu'il m'a appris en Latex mais aussi en musique « moderne ». Un grand merci à Svetlana pour ses encouragements, son soutien attentif et tous les moments partagés ensemble.

Je remercie mes anciens binômes Bruno P. et Thierry M. pour le temps passé à programmer ou partager notre passion pour le piano ou le tennis. Je remercie également mes trois fidèles bandits du modèle de Black-Scholes : Noufel, Joachim et Ludovic (qui a supporté mon amour fulgurant pour les options américaines). Merci à Jérôme et Pierrick pour être là depuis si longtemps... Merci enfin à Tom Y., Chris M., Dave G., Tom S., Mike S. et Matthew B. de m'avoir accompagné pendant toutes ces années, au bout de la nuit comme au bout du monde, par exemple pendant la rédaction du manuscrit.

Je tiens enfin à remercier ma famille et surtout mes parents pour leur soutien indéfectible, leurs sacrifices, leurs encouragements constants, leur goût pour le travail et la persévérance mais aussi tout simplement et surtout pour leur amour.



# Table des matières

<b>Introduction générale</b>	<b>1</b>
1 Motivations et résultats existants . . . . .	2
1.1 Solutions de viscosité : contexte, intérêt et quelques applications . . . . .	2
1.2 Les équations elliptiques et paraboliques non-linéaires . . . . .	3
1.3 Liens entre jeux et équations aux dérivées partielles . . . . .	6
1.3.1 Mouvement par courbure moyenne et jeu de Paul et Carol . . . . .	7
1.3.2 Les équations elliptiques et paraboliques : le jeu de Mark et Helen . . . . .	8
1.4 Théorie de la régularité elliptique et parabolique . . . . .	9
2 Les résultats de la thèse . . . . .	12
2.1 Le problème de Neumann pour les équations elliptiques et paraboliques non-linéaires	12
2.1.1 Discussion sur le choix de la dynamique, hypothèses sur le domaine et notations . . . . .	13
2.1.2 Le cas parabolique sans la dépendance en $u$ . . . . .	15
2.1.3 Le cas elliptique . . . . .	25
2.1.4 Quelques généralisations et questions ouvertes . . . . .	27
2.2 Résultats de régularité parabolique . . . . .	29
2.2.1 Théorèmes d'estimation $W^{3,\varepsilon}$ parabolique et de régularité partielle parabolique . . . . .	29
2.2.2 Eléments de la preuve de l'estimation $W^{3,\varepsilon}$ parabolique . . . . .	31
2.2.3 Preuve de l'estimation $W^{2,\varepsilon}$ parabolique . . . . .	33
2.2.4 Preuve du résultat de régularité partielle parabolique . . . . .	34
2.3 Estimation d'erreurs pour des schémas explicites et implicites pour des équations elliptiques ou paraboliques non-linéaires . . . . .	35

2.3.1	Schémas elliptiques issus du principe de programmation dynamique . . .	37
2.3.2	Quelques éléments de preuve du taux . . . . .	39
2.3.3	Schémas par différences finies pour des équations paraboliques . . . . .	41
3	Perspectives . . . . .	44

<p><b>Chapter 1</b></p> <p><b>A game interpretation of the Neumann problem for fully nonlinear parabolic and elliptic equations</b></p>
---

1.1	Introduction . . . . .	48
1.2	The games . . . . .	52
1.2.1	The linear heat equation . . . . .	52
1.2.2	General parabolic equations . . . . .	55
1.2.2.1	Heuristic derivation of the optimal player strategies . . . . .	59
1.2.2.2	Main parabolic result . . . . .	66
1.2.3	Nonlinear elliptic equations . . . . .	66
1.3	Convergence . . . . .	70
1.3.1	Viscosity solutions with Neumann condition . . . . .	70
1.3.2	The parabolic case . . . . .	72
1.3.3	The elliptic case . . . . .	77
1.4	Consistency . . . . .	78
1.4.1	The parabolic case . . . . .	79
1.4.1.1	Preliminary geometric lemmas . . . . .	79
1.4.1.2	Consistency estimates . . . . .	82
1.4.2	Proof of Lemma 1.4.9 and Proposition 1.4.10 . . . . .	86
1.4.2.1	Proof of Lemma 1.4.9 . . . . .	86
1.4.2.2	Proof of Proposition 1.4.10 case (i) . . . . .	87
1.4.2.3	Proof of Proposition 1.4.10 case (ii) . . . . .	88
1.4.2.4	Proof of Proposition 1.4.10 case (iii) . . . . .	89
1.4.2.5	Proof of Proposition 1.4.10 case (iv) . . . . .	89
1.4.3	Application to stability . . . . .	94
1.4.4	The elliptic case . . . . .	96

---

1.5	Stability . . . . .	99
1.5.1	The parabolic case . . . . .	99
1.5.2	The elliptic case . . . . .	100
1.6	Some natural generalizations . . . . .	103
1.6.1	Elliptic PDE with mixed Dirichlet-Neumann boundary conditions . . . . .	103
1.6.2	Parabolic PDE with an oblique boundary condition . . . . .	104

## Chapter 2

### Quadratic expansions and partial regularity for nonlinear parabolic equations

2.1	Introduction . . . . .	107
2.1.1	Motivation and statement . . . . .	107
2.2	Preliminaries and proof of the partial regularity result . . . . .	109
2.2.1	Hypotheses and notation . . . . .	109
2.2.2	Preliminaries . . . . .	110
2.2.3	Proof of Theorem 2.1.1 . . . . .	111
2.3	Parabolic $W^{2,\varepsilon}$ estimate . . . . .	113
2.4	Proof of Theorem 2.1.2 . . . . .	130

## Chapter 3

### Rates of convergence for elliptic dynamic programming schemes

3.1	Introduction . . . . .	140
3.1.1	Motivation and statement of the result . . . . .	140
3.1.2	Hypotheses and notation . . . . .	141
3.2	Proof of the rate of convergence . . . . .	144
3.2.1	Rate near the boundary . . . . .	144
3.2.2	Rate far the boundary . . . . .	145
3.3	The dynamic programming schemes : presentation and properties . . . . .	148
3.3.1	Presentation of the games . . . . .	148
3.3.2	Proof of the assumptions (S1) and (S2) on $F_\varepsilon$ . . . . .	150
3.3.3	Proof of Proposition 3.1.2 . . . . .	151
3.3.3.1	Continuity of the score functions . . . . .	151



3.3.3.2	Consistency of the scheme with the boundary condition . . . . .	155
3.4	Proof of Proposition 3.2.2 . . . . .	161
3.5	Appendix . . . . .	166

<b>Chapter 4</b> <b>Rates of convergence for parabolic finite-difference schemes</b>
---

4.1	Introduction . . . . .	168
4.1.1	Motivation and statement of the results . . . . .	168
4.1.2	Hypotheses and notation . . . . .	169
4.2	Proof of the implicit rate of convergence . . . . .	171
4.2.1	The $W^{3,\sigma}$ estimate . . . . .	171
4.2.2	A comparison principle for implicit schemes . . . . .	172
4.2.3	A maximum principle in the doubled variables . . . . .	173
4.2.4	Proof of the main theorem . . . . .	173
4.3	Proof of the maximum principle in the doubled variables . . . . .	182
4.4	Appendix A: continuous version of the doubling lemma . . . . .	191
4.5	Appendix B: Proof of Proposition 4.2.5 . . . . .	196

<b>Annexes</b>
----------------

<b>Annexe A</b> <b>Estimation <math>W^{3,\varepsilon}</math> elliptique</b>
--

A.1	Introduction . . . . .	202
A.2	Proof of Theorem A.1.1 . . . . .	202
A.3	Proof of Proposition A.2.1 . . . . .	204

<b>Annexe B</b> <b>Poster</b>
----------------------------------

<b>Bibliographie</b>
----------------------

<b>Résumé</b>	<b>223</b>
<b>Abstract</b>	<b>224</b>

# Introduction générale



# Introduction générale

Cette thèse est consacrée à des résultats d'approximation et de régularité pour des équations elliptiques et paraboliques non-linéaires étudiées dans le cadre des solutions de viscosité. L'étude des solutions de viscosité de ces équations est particulièrement motivée par :

- ◊ La théorie des *jeux différentiels*, qui fait intervenir à la fois la théorie des jeux (plusieurs joueurs interviennent) et des systèmes contrôlés (les équations différentielles sont contrôlées par les joueurs),
- ◊ La théorie du contrôle optimal déterministe et stochastique,
- ◊ La théorie des propagations d'interface et de l'évolution de surfaces.

Elle peut être abordée sous des angles variés, parmi lesquels :

- ◊ Elaboration d'une théorie de la régularité,
- ◊ Approximation et représentation de la solution de viscosité,
- ◊ Homogénéisation de problèmes déterministes et stochastiques.

La présente introduction se divise en trois parties. Dans une première partie, nous donnerons les motivations et le cadre du travail effectué dans la thèse. Les résultats obtenus sont ensuite rassemblés dans une seconde partie où nous les commentons et donnons certaines idées de leurs preuves. Enfin, une troisième partie est consacrée à certaines perspectives possibles.

Avant de commencer, donnons quelques notations qui seront utilisées au cours de l'introduction.

## Quelques notations :

- $\langle x, y \rangle$  est le produit scalaire usuel entre deux vecteurs  $x$  et  $y$  de  $\mathbb{R}^d$  et  $|\cdot|$  représente la norme euclidienne associée.
- $B_r(x) \subseteq \mathbb{R}^d$  est la boule fermée de centre  $x \in \mathbb{R}^d$  et de rayon  $r > 0$ .
- $Q_r(x, t) \subseteq \mathbb{R}^{d+1}$  est le cylindre parabolique de centre  $(x, t) \in \mathbb{R}^{d+1}$  et de rayon  $r > 0$  donné par  $Q_r(x, t) := B_r(x) \times ]t - r^2, t]$ .
- Si  $\mathcal{O}$  est un ensemble ouvert de  $\mathbb{R}^d$  et  $v : \mathcal{O} \rightarrow \mathbb{R}$ , alors  $Dv(x)$  et  $D^2v(x)$  représentent respectivement le gradient et la matrice hessienne de  $v$  au point  $x$  lorsqu'on peut leur donner un sens. De la même façon, si  $\mathcal{O}$  est un ensemble ouvert de  $\mathbb{R}^{d+1}$  et  $v = v(x, t) : \mathcal{O} \rightarrow \mathbb{R}$  dépend du temps et de l'espace, on note  $\partial_t v(x, t)$  la dérivée temporelle de  $v$  et par  $Dv(x, t)$  et  $D^2v(x, t)$  le gradient et la matrice hessienne de  $v$  au point  $(x, t)$ .
- Une fonction semi-continue inférieurement (resp. supérieurement) sera notée SCI (resp. SCS).
- $\mathcal{M}_d(\mathbb{R})$  est l'espace des matrices réelles  $d \times d$ ,  $\mathcal{S}_d \subseteq \mathcal{M}_d(\mathbb{R})$  l'ensemble des matrices symétriques de taille  $d \times d$ , et  $I_d$  la matrice identité. La trace de  $M$  sera notée par  $\text{tr } M$ , son déterminant  $\det M$ , son adjoint  $M^\top$  et sa norme de Hilbert-Schmidt  $\sqrt{M^\top M}$  par  $\|M\|$ .
- Une constante sera dite *universelle* si elle ne dépend que de la dimension et des constantes d'ellipticité.

# 1 Motivations et résultats existants

## 1.1 Solutions de viscosité : contexte, intérêt et quelques applications

L'étude au sens de viscosité des solutions d'une équation non-linéaire constitue une alternative aux distributions pour définir des solutions faibles. Cette approche basée sur le principe du maximum propose une définition ponctuelle de la solution et utilise des espaces fonctionnels formés de fonctions semi-continues inférieurement et supérieurement. Ce cadre est particulièrement pertinent lorsque la non-linéarité ne se présente pas sous forme divergence, c'est-à-dire lorsque sa structure ne se prête pas à des intégrations par parties. La plupart des équations étudiées par ces techniques proviennent de la propagation d'interfaces, de la théorie du contrôle optimal ou de celle des jeux différentiels, ces deux derniers champs faisant intervenir tous les deux la notion de contrôle.

L'idée de *contrôle* peut être décrite comme le processus d'influencer le comportement d'un système dynamique en vue de réaliser un certain objectif. Si nous nous donnons pour but d'optimiser une certaine fonction coût (ou "payoff") qui dépend des données de contrôle du système, le problème relève alors du *contrôle optimal*. Dans les années 1950 et 1960, les applications en ingénierie aérospatiale ont largement stimulé le développement de la théorie du contrôle optimal déterministe. Nous pouvons notamment citer parmi elles les trajectoires optimales de vol pour les avions et les véhicules spatiaux. Cependant, la théorie du contrôle déterministe fournit des méthodes dont l'application est beaucoup plus étendue, s'étendant à des problèmes issus de différents domaines comme l'ingénierie, l'économie ou la science du management. En faisant intervenir des modèles probabilistes (chaînes de Markov, équations différentielles stochastiques backward ou classiques, processus à saut, etc.), il est aussi possible d'étudier des problèmes de contrôle stochastique (voir les ouvrages [68, 147] et aussi [122]).

Le principe de programmation dynamique est une méthode classique pour étudier ces problèmes déterministes ou stochastiques, initiée dès les années 1950 par Bellman. Son idée fondamentale a été de considérer une famille de problèmes de contrôle en faisant varier les valeurs des états initiaux et d'obtenir des relations entre les fonctions valeurs associées. Elle permet de dériver formellement une certaine équation aux dérivées partielles (EDP) non-linéaire, appelée équation d'Hamilton-Jacobi Bellman (HJB). Pour ce type de problèmes, la question de l'existence de solutions ne se pose pas en général car nous savons a priori que la fonction-valeur doit être solution de l'équation. Au contraire, nous souhaitons caractériser cette fonction valeur et nous nous intéressons par exemple à sa stabilité, à son approximation numérique ou à sa régularité. Lorsque cette EDP peut être résolue par la réalisation explicite ou théorique d'une fonction régulière, le théorème de vérification valide l'optimalité de la solution candidate à l'équation d'HJB. Cette approche classique pour l'équation de programmation dynamique est appelée l'étape de vérification. Le principal inconvénient de cette approche est de supposer l'existence *a priori* d'une fonction valeur régulière solution de l'équation d'HJB. Cela n'est pas nécessairement vrai même dans des cas très simples.

Pour résoudre cette difficulté, Crandall et Lions [48] (voir aussi [47]) et Evans [58, 59] ont développé simultanément un concept de solutions faibles pour les équations linéaires ou non-linéaires qui ne sont pas sous forme divergence. Il remplace le principe de Dirichlet et le concept de solutions variationnelles dans la théorie des équations sous forme divergence. Crandall et Lions ont introduit dans un article célèbre de 1983 une formulation faible pour les équations du premier ordre qu'ils ont appelé *solutions de viscosité*. Bien que le mot "viscosité" se réfère à un schéma de relaxation (consistant à introduire un terme  $\varepsilon \Delta u$  de diffusion avec un paramètre  $\varepsilon \rightarrow 0$ ), la définition de solution de viscosité est intrinsèque. En particulier, les solutions de viscosité sont stables sous une relaxation raisonnable ou une approximation raisonnable de l'équation. Un résultat d'unicité pour les équations du premier ordre a été une autre contribution très importante de cet article (voir [44] pour d'autres formulations équivalentes des solutions de viscosité). Cela explique pourquoi les principaux résultats de la théorie des solutions de viscosité sont des résultats d'unicité et de passage à la limite, ces deux questions étant en fait indissociables. Les résultats d'unicité sont toujours obtenus par des "théorèmes de comparaison", dont les techniques de démonstration constituent le coeur

de la théorie. Les solutions de viscosité fournissent des moyens très puissants pour étudier en grande généralité des problèmes de contrôle et donnent une formulation rigoureuse de l'équation d'HJB pour des fonctions supposées seulement localement bornées. En combinant ces résultats avec des principes de comparaison pour les solutions de viscosité, la fonction valeur est alors identifiée et caractérisée comme étant l'unique solution de viscosité de l'équation de programmation dynamique associée. Cela peut ensuite être éventuellement utilisé pour obtenir des résultats plus avancés.

Cette théorie a été ensuite généralisée aux équations du second ordre. Ces équations seront présentées à la section 0.1.2. Par ailleurs, l'étude de la régularité des solutions de viscosité a été essentiellement effectuée par Caffarelli en utilisant des techniques dues principalement à Jensen [86] en 1988. Elle sera revisitée à la section 0.1.4.

Les équations concernées interviennent dans la géométrie, le traitement d'images, la propagation de flammes en écoulements turbulents, la théorie des transitions de phase en mécanique des milieux continus, etc. Par exemple, Alvarez, Guichard, Lions et Morel [1] ont étudié l'analyse multi-échelle d'une image par des outils de viscosité. La propagation de fronts, notamment pour la courbure moyenne a donné lieu à de nombreux travaux, notamment ceux de Chen, Giga, Goto [41] (voir aussi [72]), ou de Barles, Soner et Souganidis [21, 129, 134, 135]. Il existe également des connexions entre la propagation de fronts et les jeux différentiels (voir [131]).

## 1.2 Les équations elliptiques et paraboliques non-linéaires

Nous allons nous intéresser principalement aux solutions de viscosité d'équation d'ordre 1 ou 2 dites "elliptiques" ou "paraboliques". Les équations elliptiques ont la forme générale

$$\mathcal{F}(x, u(x), Du(x), D^2u(x)) = 0, \quad \text{pour } x \in U \subseteq \mathbb{R}^d,$$

où  $u : U \rightarrow \mathbb{R}$  est l'inconnue, et  $\mathcal{F} : U \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  est une application donnée ( $\mathcal{S}_d$  désignant l'ensemble des matrices symétriques de taille  $d \times d$ ). Nous supposons systématiquement par la suite que l'équation est *elliptique*, ce qui signifie que

(E) (**Ellipticité**) La non-linéarité  $\mathcal{F} = \mathcal{F}(x, s, p, M)$  est monotone décroissante par rapport à la matrice  $M$ , i.e. si pour tout  $(x, r, p) \in \bar{U} \times \mathbb{R} \times \mathbb{R}^d$ , nous avons

$$\mathcal{F}(x, r, p, M) \leq \mathcal{F}(x, r, p, N) \quad \text{pour tout } M, N \in \mathcal{S}_d \text{ tel que } M \geq N,$$

où  $\geq$  est la relation d'ordre partiel pour  $\mathcal{S}_d$  :  $M \geq N$  signifie que  $M - N$  est une matrice semi-définie positive.

Dans le cas parabolique, l'équation générale prend la forme

$$\partial_t u(x, t) + \mathcal{F}(t, x, u(x, t), Du(x, t), D^2u(x, t)) = 0 \quad \text{pour } (x, t) \in \Omega$$

où  $\Omega \subseteq \mathbb{R}^{d+1}$  est ouvert,  $u : \Omega \rightarrow \mathbb{R}$  est l'inconnue, et  $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \rightarrow \mathbb{R}$  est décroissante par rapport à  $M$  ( $\mathcal{F}$  sera dite elliptique).

En l'absence d'hypothèse supplémentaire, en particulier d'uniforme ellipticité, l'équation elliptique ou parabolique peut être dégénérée : nous ne supposons pas en général que  $\mathcal{F}$  est "strictement" décroissante par rapport à  $M$ . En particulier,  $\mathcal{F}$  peut très bien être complètement indépendante de  $M$ , et l'équation est dans ce cas d'ordre 1.

Par souci de simplicité et sans perte de généralité, nous ne considérerons pas toujours l'opérateur  $\mathcal{F}$  général avec toutes les dépendances possibles en  $(t, x, u, Du, D^2u)$ , mais nous nous concentrerons parfois sur une sous-classe d'opérateurs elliptiques dépendant seulement de la hessienne  $D^2u$ . Nous considérerons alors des équations typiquement de la forme

$$\mathcal{F}(D^2u(x)) = g(x) \quad \text{pour } x \in U \subseteq \mathbb{R}^d.$$

ou

$$\partial_t u(x, t) + \mathcal{F}(D^2 u(x, t)) = g(x, t) \quad \text{pour } (x, t) \in \Omega \subseteq \mathbb{R}^{d+1}.$$

Dans ce cadre, nous allons introduire des conditions sur la forme de l'opérateur mais elles ne sont pas véritablement restrictives. Nous donnerons ensuite rapidement les généralisations possibles. Tout d'abord, nous aurons parfois besoin d'imposer la condition plus forte d'*uniforme ellipticité* donnée par :

- (F1) **(Uniforme ellipticité)**  $\mathcal{F}$  est uniformément elliptique et lipschitzienne ; précisément, nous supposons qu'il existe deux constantes  $0 < \lambda \leq \Lambda$  telles que, pour tout  $M, N \in \mathcal{S}_d$ ,

$$\mathcal{P}_{\lambda, \Lambda}^-(M - N) \leq \mathcal{F}(M) - \mathcal{F}(N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M - N).$$

où  $\mathcal{P}_{\lambda, \Lambda}^-$  et  $\mathcal{P}_{\lambda, \Lambda}^+$  sont les opérateurs extrémaux de Pucci donnés par

$$\mathcal{P}_{\lambda, \Lambda}^+(M) := \sup_{\lambda I_d \leq A \leq \Lambda I_d} -\text{tr}(AM), \quad \text{et} \quad \mathcal{P}_{\lambda, \Lambda}^-(M) := \inf_{\lambda I_d \leq A \leq \Lambda I_d} -\text{tr}(AM). \quad (1)$$

Nous pourrions éventuellement demander une régularité supplémentaire donnée par :

- (F2) **(Régularité de  $D\mathcal{F}$ )**  $\mathcal{F}$  est  $C^1$  et sa dérivée  $D\mathcal{F}$  est uniformément continue, c'est-à-dire, il existe une fonction croissante continue  $\omega : [0, \infty) \rightarrow [0, \infty)$  telle que  $\omega(0) = 0$  et pour tout  $M, N \in \mathcal{S}_d$ ,

$$|D\mathcal{F}(M) - D\mathcal{F}(N)| \leq \omega(|M - N|).$$

L'hypothèse (F1) d'uniforme ellipticité s'interprète comme une propriété de stabilité : elle fournit un contrôle sur la perturbation autour d'une matrice fixée. En effet, l'opérateur  $\mathcal{F}$  vérifie alors :

$$\mathcal{F}(M) - d\Lambda \|N\| \leq \mathcal{F}(M + N) \leq \mathcal{F}(M) - d\lambda \|N\| \quad \text{pour tout } M, N \in \mathcal{S}_d, \quad N \geq 0. \quad (2)$$

Nous imposons un contrôle sur la non-linéarité sous l'effet d'une perturbation positive : elle est strictement décroissante (rôle de  $\lambda$ ) mais elle ne peut pas être trop grande (rôle de  $\Lambda$ ). Nous pouvons aussi noter que les deux formulations (F1) et (2) sont équivalentes quitte à changer les constantes d'ellipticité et peuvent donc définir de façon équivalente les opérateurs uniformément elliptiques.

**Remarque 1.** Si nous ajoutons des dépendances supplémentaires, l'hypothèse d'uniforme ellipticité doit être remplacée par pour tout  $(x, t) \in \Omega$

$$\begin{aligned} \mathcal{P}_{\lambda, \Lambda}^-(M - N) - \gamma|p - q| - \omega(|s - r|) &\leq \mathcal{F}(x, t, r, p, M) - \mathcal{F}(x, t, s, q, N) \\ &\leq \mathcal{P}_{\lambda, \Lambda}^+(M - N) + \gamma|p - q| + \omega(|s - r|). \end{aligned}$$

pour  $M, N \in \mathcal{S}_d$ ,  $p, q \in \mathbb{R}^d$  et  $r, s \in \mathbb{R}$ . Pour la dépendance en  $u$ , il est alors souvent nécessaire d'introduire une hypothèse de croissance en la variable  $u$  afin d'assurer un principe de comparaison sur l'équation.

Nous poursuivons cette introduction par un bref panorama de quelques exemples des équations elliptiques et paraboliques non-linéaires que nous avons en vue. Les exemples les plus simples sont évidemment les équations linéaires sans second membre que sont l'équation de Laplace et de la chaleur données respectivement par

$$-\Delta u = - \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = 0$$

et

$$\partial_t u - \Delta u = 0.$$

Nous allons maintenant présenter des équations non-linéaires.

*Exemple 1 : les équations de Pucci.* Elles sont de la forme

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2 u) = g(x) \quad \text{ou} \quad \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) = g(x) \quad \text{pour } x \in U,$$

où  $\mathcal{P}_{\lambda,\Lambda}^-$  et  $\mathcal{P}_{\lambda,\Lambda}^+$  sont les opérateurs extrémaux de Pucci définis par (1). On peut aussi les écrire de manière explicite

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = -\lambda \sum_{\mu_j > 0} \mu_j - \Lambda \sum_{\mu_j < 0} \mu_j \quad \text{et} \quad \mathcal{P}_{\lambda,\Lambda}^-(M) = -\Lambda \sum_{\mu_j > 0} \mu_j - \lambda \sum_{\mu_j < 0} \mu_j,$$

où  $\mu_1, \dots, \mu_d$  sont les valeurs propres de  $M \in \mathcal{S}_d$ . Ils constituent des opérateurs de référence car ils permettent de définir et d'étudier la classe générale des opérateurs uniformément elliptiques définis par (F1). Ils sont eux-mêmes uniformément elliptiques et non-linéaires. Des manipulations algébriques donnent les inégalités

$$\begin{aligned} \mathcal{P}_{\lambda,\Lambda}^-(M) + \mathcal{P}_{\lambda,\Lambda}^-(N) &\leq \mathcal{P}_{\lambda,\Lambda}^-(M+N) \leq \mathcal{P}_{\lambda,\Lambda}^-(M) + \mathcal{P}_{\lambda,\Lambda}^+(N) \\ &\leq \mathcal{P}_{\lambda,\Lambda}^+(M+N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M) + \mathcal{P}_{\lambda,\Lambda}^+(N). \end{aligned}$$

De plus,  $\mathcal{P}_{\lambda,\Lambda}^-$  et  $\mathcal{P}_{\lambda,\Lambda}^+$  sont respectivement concave et convexe. L'un des intérêts majeurs pour introduire les opérateurs de Pucci est le suivant : si  $u$  est solution d'un problème uniformément elliptique de la forme  $\mathcal{F}(D^2u) = g(x)$ , alors  $u$  est à la fois une sur- et une sous-solution de viscosité d'équations qui ne dépendent plus de  $\mathcal{F}$ , mais uniquement des constantes  $\lambda$ ,  $\Lambda$  et de  $g$  car elle vérifie

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2u) + |g(x)| + |\mathcal{F}(0)| \geq 0 \quad \text{et} \quad \mathcal{P}_{\lambda,\Lambda}^-(D^2u) - |g(x)| - |\mathcal{F}(0)| \leq 0.$$

*Exemple 2 : les équations de Bellman et d'Isaacs.* Lorsque la famille  $\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathcal{S}_d : \lambda I_d \leq A \leq \Lambda I_d\}$  utilisée dans (1) pour définir les opérateurs extrémaux de Pucci est remplacée par une famille arbitraire  $\mathcal{A}$  d'opérateurs elliptiques linéaires, nous obtenons l'équation de Bellman pour le coût optimal dans un problème de contrôle stochastique (voir par exemple [89]). Plus précisément, l'équation de Bellman prend la forme

$$\mathcal{F}(D^2u, x) = \inf_{\alpha \in \mathcal{A}} (L_\alpha u(x) - g_\alpha(x)) = 0 \quad \text{pour } x \in U,$$

où  $\mathcal{A}$  est un ensemble fixé,  $g_\alpha : U \rightarrow \mathbb{R}$  pour tout  $\alpha \in \mathcal{A}$ , et  $L_\alpha u = a_\alpha^{ij}(x) \partial_{ij} u$  est un opérateur uniformément elliptique avec des coefficients mesurables bornés (voir [118]).

Les équations d'Isaacs apparaissent naturellement dans le contexte des jeux différentiels (voir [117]) et sont de la forme

$$\mathcal{F}(D^2u, x) := \sup_\beta \inf_\alpha (L_{\alpha\beta} u(x) - g_{\alpha\beta}(x)) = 0 \quad \text{pour } x \in U,$$

où  $L_{\alpha\beta}$  est une famille d'opérateurs elliptiques avec des coefficients mesurables bornés, de constantes d'ellipticité  $\lambda$  et  $\Lambda$ , et indexée par  $\alpha$  et  $\beta$  appartenant à deux ensembles arbitraires. Les équations d'Isaacs sont toujours uniformément elliptiques, mais ne sont ni concaves ni convexes.

*Exemple 3 : le mouvement par courbure moyenne.* Cette équation est le prototype d'équation dégénérée et singulière. Elle s'écrit dans sa version elliptique par :

$$\begin{cases} -\Delta u + \frac{\langle D^2u Du, Du \rangle}{|Du|^2} = -1, & \text{sur } U \\ u = 0, & \text{sur } \partial U. \end{cases}$$

C'est bien une équation qui satisfait la condition (E) d'ellipticité, avec

$$\mathcal{F}(p, M) = -\text{tr } M + \frac{\langle Mp, p \rangle}{|p|}.$$

Cette équation est singulière en  $p = 0$  et dégénérée car pour tout  $(p, M) \in \mathbb{R}^d \setminus \{0\} \times \mathcal{S}_d$ ,

$$\mathcal{F}(p, M + \mu p p^\top) = \mathcal{F}(p, M).$$



L'étude des solutions de viscosité a été menée par Evans et Spruck dans une série de quatre articles [63, 64, 65, 66]. Ils donnent notamment une définition des solutions de viscosité pour cette équation qui est adaptée au traitement de la singularité en  $p = 0$ . Barles et Georgelin [14] ont ensuite proposé d'autres définitions équivalentes. Plus largement, la littérature concernant le mouvement par courbure moyenne est très riche. Pour l'étude particulière de l'épaississement des lignes de niveaux ("fattening") pour le mouvement à courbure moyenne, nous pouvons citer [22].

*Exemple 4 : l'équation de Monge-Ampère.* Elle intervient dans plusieurs problèmes en analyse et en géométrie. Elle provient d'un problème de transport formulé initialement en 1781 par Monge [112] et qui a ensuite engendré un très grand nombre de travaux, notamment ces dernières années. La question posée était : comment déplacer, dans l'espace  $\mathbb{R}^3$  muni de sa structure euclidienne, un déblai pour construire un remblai ou fortification tout en minimisant un coût de transport ? La forme classique de l'équation est

$$\det D^2u = g(x, u, Du) \quad \text{dans } U \subseteq \mathbb{R}^d \text{ ouvert,} \quad (3)$$

où  $u : U \rightarrow \mathbb{R}$  est strictement convexe et  $g : U \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}_+^*$  est donnée.

Dans ce cas,  $\mathcal{F}(M) = \det M$  et  $\mathcal{F}_{ij}(M) := \frac{\partial \mathcal{F}}{\partial m_{ij}}(M)$  est le cofacteur de  $M_{ij}$ . Si la matrice  $M$  est inversible,  $\mathcal{F}_{ij}(M) = (\det M)(M^{-1})_{ij}$ . Les conditions de stricte convexité sur  $u$  et la stricte positivité de  $g$  sont alors suffisantes pour assurer respectivement l'ellipticité et l'existence d'une solution de (3). Même si l'opérateur de Monge-Ampère n'est pas uniformément elliptique sur l'ensemble des matrices symétriques tout entier, il est possible d'adapter de nombreuses méthodes pour cette équation (voir [32, 33] et aussi [30] ainsi que des travaux récents de Figalli, De Philippis et Savin [52, 53, 55] sur la régularité des solutions). Pour des liens plus spécifiques avec le transport optimal, nous renvoyons le lecteur au récent panorama dressé par Figalli et De Philippis [54] et la monographie [142] de Villani (voir aussi [75] pour une revue globale).

Les équations elliptiques que l'on vient de présenter possèdent une version **parabolique**, qui est obtenue généralement en additionnant un terme de dérivée temporelle devant la non-linéarité. Cependant, en raison de l'interprétation en termes de volume par le jacobien, la version parabolique usuelle de l'équation de Monge-Ampère s'écrit

$$\partial_t u \det D^2u = g(x, u, Du) \quad \text{sur } \Omega.$$

D'autres équations paraboliques de Monge-Ampère peuvent également être étudiées. Par exemple, Ivchikina et Ladyzhenskaya ont étudié

$$\partial_t u + (\det D^2u)^{1/d} = g(x, u, Du) \quad \text{sur } \Omega,$$

ainsi que des variantes dans [80, 81, 82, 83].

### 1.3 Liens entre jeux et équations aux dérivées partielles

Dans les problèmes de contrôle optimal ou issus de la théorie des jeux que nous avons en vue, le problème peut être décrit de la façon générique suivante :

- Nous considérons un certain nombre  $N = 1, 2, \dots, +\infty$  d'agents (joueurs, individus, etc. suivant la modélisation adoptée) qui souhaitent optimiser un certain critère, c'est-à-dire en termes plus mathématiques, minimiser/maximiser une certaine fonctionnelle  $J$ . Suivant le domaine concerné : ingénierie, économie, finance, ce critère est appelé utilité (portefeuille de Markowitz), profit (économie), coût (ingénierie), score (théorie des jeux), etc.
- Cette fonctionnelle  $J$  dépend de la dynamique d'un processus, qui peut être par exemple le cours d'une action, le mouvement d'une particule. Cette dynamique peut être continue ou discrète, déterministe ou stochastique.
- Les agents peuvent influencer sur la dynamique du processus via un paramètre de contrôle et prennent des décisions rationnelles. Cela consiste à dire que le paramètre de contrôle, choisi par l'agent, sera ajusté de sorte à maximiser le résultat du joueur vis-à-vis de la fonctionnelle  $J$ .

Prenons pour exemple préliminaire les jeux différentiels s'écrivant sous la forme d'un système dynamique contrôlé avec condition initiale :

$$\begin{cases} \frac{dy(s)}{ds} = f(y(s), \alpha(s)), \\ y(0) = x, \end{cases}$$

où  $x \in \mathbb{R}^d$  est la condition initiale,  $f$  est continue et Lipschitz par rapport à  $\alpha(\cdot) \in \mathcal{A}$  où  $\mathcal{A}$  est l'ensemble de contrôle qui possède généralement la structure d'un espace métrique compact. Evans et Souganidis [62] ont proposé une formule de représentation pour des EDP du premier ordre en mettant un jeu des jeux différentiels. D'autres résultats dans cette direction ont ensuite été produits dans la même direction. Il existe aussi des liens avec les équations différentielles stochastiques rétrogrades (voir [118, 67]).

Dans la suite de ce paragraphe, nous allons nous intéresser à des jeux qui permettent de représenter la solution de certaines équations non-linéaires. Plus précisément, la situation standard est de construire une famille  $(G_\varepsilon)_\varepsilon$  de jeux et de s'intéresser à l'équation formelle

$$\ll u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon \gg,$$

où  $u$  est la solution de viscosité de l'EDP et  $u^\varepsilon$  est le score donné par le jeu pour  $\varepsilon > 0$  fixé. Le but est alors de rendre rigoureux la convergence (dans un certain sens) de la suite  $u^\varepsilon$  des scores vers la solution de viscosité  $u$  de l'EDP. On peut faire tout de suite deux commentaires sur ce genre de résultats de convergence :

- Le jeu fournit une formule asymptotique de représentation de la solution de l'EDP, qui fournit donc à  $\varepsilon$  fixé une approximation de cette solution.
- La convergence est généralement obtenue uniformément sur des compacts mais elle n'est pas quantifiée directement ou a priori par un taux de convergence. Cela fera l'objet de la section 0.2.3.

La table 1 propose une classification non-exhaustive des équations et de leurs jeux associés. Peres, Schramm, Sheffield et Wilson [121] ont proposé l'interprétation de l'équation du laplacien infini par le jeu du "tug-of-war" ou *remorqueur de guerre* par des outils à la fois d'EDP et de probabilités. Armstrong et Smart [3] ont ensuite étudié ce jeu par des méthodes presque exclusivement EDP comme le principe de Perron. Serfaty et Imbert [77] ont donné une interprétation par jeux répétés pour certaines équations non-locales comme l'équation eikonale ou le laplacien fractionnaire. Nous pouvons enfin signaler que Caffarelli, Figalli et Bjorland ont proposé des versions non-locales du jeu du "tug-of-war" (voir [24] et [23]).

Equation	Jeu	Dynamique
Hamiltonien du 1 <sup>er</sup> ordre	différentiel [62]	déterministe
Laplacien infini	Tug-of-war [121]	stochastique
Mouvement par courbure moyenne	Paul et Carol [87]	déterministe
EDP elliptiques et paraboliques non-linéaires	Mark et Helen [88]	déterministe
EDP non-locales	Mark et Helen non local [77]	déterministe
	Tug-of-war non local [24]	stochastique

Tableau 1 – EDP, jeux et dynamiques associés.

### 1.3.1 Mouvement par courbure moyenne et jeu de Paul et Carol

Le jeu de Paul et Carol, associé au mouvement par courbure moyenne, a été inventé par Joel Spencer [136]. Kohn et Serfaty [87] ont ensuite proposé et étudié rigoureusement la correspondance adaptée entre le jeu et l'EDP pour le mouvement par courbure moyenne en dimension  $d = 2$  (qu'il est alors aisé de généraliser à des dimensions  $d \geq 3$ ). Ce jeu fait intervenir deux joueurs, Paul et Carol, qui prennent

toujours des décisions rationnelles et déterministes. Les règles dépendent d'un petit paramètre  $\varepsilon$ , qui gouverne l'échelle spatiale et, pour les problèmes dépendant du temps, le pas de temps. Paul démarre de la position  $x \in U$  et veut sortir du domaine  $U$  dans un temps minimal, i.e. en procédant avec un nombre minimum de coups. Carol est son opposant et veut donc le bloquer. Plus précisément, soit  $\varepsilon > 0$  est donné. Les étapes du jeu sont les suivantes :

**Etape 1 :** Paul choisit un vecteur  $v$  de norme 1.

**Etape 2 :** Carol choisit un nombre  $b = \pm 1$  et la particule se déplace de la position actuelle jusqu'à la nouvelle position

$$x_{k+1} = x_k + \sqrt{2}\varepsilon bv.$$

**Etape 3 :** Le jeu se termine si  $x^* = x_{k+1} \notin U$ , et la fonction coût est

$$t^* = \text{instant où le processus sort de } U.$$

Si ce n'est pas le cas, les étapes précédentes sont répétées à partir de la nouvelle position  $x_{k+1}$ , jusqu'à la sortie.

Autrement dit, Paul veut sortir de  $U$  le plus vite possible tandis que Carol veut l'empêcher aussi longtemps que cela est possible.

Si  $u^\varepsilon(x)$  représente le payoff, étant donné qu'on commence en  $x = x_0$ , le principe de programmation dynamique s'écrit :

$$u^\varepsilon(x) = \min_{|v|=1} \max_{b=\pm 1} \left( \varepsilon^2 + u^\varepsilon(x + \sqrt{2}\varepsilon bv) \right).$$

Il est important de noter que le jeu de Paul et Carol possède une interprétation géométrique claire en termes de ligne de niveau car les déplacements choisis par Paul dépendent explicitement du gradient de  $u$ . En particulier, cela permet d'étudier certains phénomènes géométriques comme l'épaississement des lignes de niveau, appelé aussi "fattening" dans la littérature. Ce phénomène a été étudié pour le mouvement par courbure moyenne via le jeu de Paul et Carol par Liu dans [110]. De plus, Sesum a étudié la régularité suivant la dimension du mouvement par courbure moyenne par des outils de géométrie différentielle [128].

Par ailleurs, il est possible de donner des interprétation stochastiques au mouvement par courbure moyenne. On peut citer les travaux de Soner et Touzi [130] et Buckdahn, Cardaliaguet et Quincampoix [26].

### 1.3.2 Les équations elliptiques et paraboliques : le jeu de Mark et Helen

Nous reviendrons en détail sur ce jeu à la section 0.2.1 de l'introduction consacrée à l'interprétation par jeux d'équations elliptiques et paraboliques non-linéaires munies d'une condition au bord de Neumann. Kohn et Serfaty [88] ont donné une interprétation par jeux d'une large classe d'EDP paraboliques et elliptiques non-linéaires respectivement de la forme

$$-\partial_t u + \mathcal{F}(t, x, u, Du, D^2u) = 0$$

et

$$\lambda u + \mathcal{F}(x, u, Du, D^2u) = 0.$$

L'équation parabolique est considérée dans l'espace  $\mathbb{R}^d \times (-\infty, T]$  et munie d'une condition finale  $u(x, T)$  tandis que l'équation elliptique est munie d'une condition de Dirichlet. Kohn et Serfaty construisent une approximation par différences de l'opérateur déduit du principe de programmation dynamique associé au jeu et montrent la consistance de cet opérateur qui possède également la propriété d'être monotone.

Plus précisément, les jeux que l'on considère font intervenir deux joueurs Mark et Helen qui prennent toujours des décisions rationnelles et déterministes. Les règles dépendent de la forme de l'équation, mais il y a toujours un petit paramètre  $\varepsilon$ , qui gouverne l'échelle spatiale et, pour les problèmes dépendant du temps, le pas de temps. Le but d'Helen est d'optimiser, ou plus exactement, de maximiser, son revenu minimal. Si l'on fait une interprétation financière :

- a) En premier lieu, nous considérons  $U^\varepsilon(x, z, t)$ , qui est la richesse optimale d'Helen à l'instant  $T$ , si initialement, à l'instant  $t$ , le prix du stock est  $x$  et sa dette est  $-z$ .
- b) Nous définissons ensuite deux fonctions  $u^\varepsilon(x, t)$  et  $v^\varepsilon(x, t)$  qui sont formellement la dette initiale qu'Helen devrait avoir à l'instant  $t$  pour que sa richesse s'annule à l'instant final  $T$ .

Lorsque  $\mathcal{F}$  n'a pas de dépendance en la variable  $u$ , la fonction valeur  $u^\varepsilon (= v^\varepsilon)$  est caractérisée par le principe de programmation dynamique. Mais si  $\mathcal{F}$  dépend de  $u$ , l'unicité de la solution de viscosité n'est plus assurée et nous avons alors besoin d'introduire une formulation level-set sur  $U^\varepsilon$  pour surmonter cette difficulté. La richesse  $U^\varepsilon$  d'Helen dépend maintenant d'un nouveau paramètre  $z \in \mathbb{R}$ . Dans le cas parabolique, la richesse vérifie une relation de récurrence rétrograde en temps donnée, pour tout  $z \in \mathbb{R}$ , par

$$U^\varepsilon(x, z, t_j) = \max_{p, \Gamma} \min_{\Delta x} U^\varepsilon(x + \Delta x, z + \Delta z, t_{j+1}) \quad (4)$$

munie de la condition finale (en temps)  $U^\varepsilon(x, z, T) = g(x) - z$ . Le max sur  $p$  et  $\Gamma$  ainsi que le min sur  $\Delta x$  sont donnés par certaines contraintes dépendant des règles du jeu et de certaines puissances de  $\varepsilon$ .

Des EDP du second ordre elliptiques ou paraboliques ont d'abord été associées à des processus stochastiques, ce qui était naturel par la formule d'Ito (voir par exemple le travail de Cheridito, Soner, Touzi et Victoir [42] qui concerne les équations backward stochastiques différentielles (BSDE) du second ordre). La représentation de solutions d'équations du second ordre par des dynamiques déterministes est seulement apparu dans l'article de Kohn et Serfaty.

A la différence du jeu de Paul et Carol pour le mouvement à courbure moyenne, les déplacements de la particule par Mark ne possèdent pas d'interprétation géométrique claire car les stratégies optimales d'Helen,  $p$  et  $\Gamma$ , vérifient la propriété remarquable que le score d'Helen devient alors indifférent au déplacement choisi par Mark. Pour toute autre valeur, Mark est vainqueur. Cette idée est similaire au raisonnement par absence d'arbitrage en mathématiques financières et en particulier : pour toute autre valeur, soit l'acheteur soit le vendeur peut faire un profit sans risque. Nous pouvons citer par exemple la couverture en delta neutre dans le modèle de Black and Scholes.

## 1.4 Théorie de la régularité elliptique et parabolique

Nous allons donner dans cette sous-section un bref aperçu des outils utilisés pour étudier la régularité des solutions de viscosité d'équations elliptiques uniformément elliptiques

$$\mathcal{F}(D^2 u) = g \quad \text{pour } U \subseteq \mathbb{R}^d, \quad (5)$$

et des résultats existants, puis nous donnerons certaines notions dans le cas parabolique.

Le cas le plus simple est celui des équations linéaires sans second membre, pour lequel nous pouvons supposer que (5) est  $\Delta u = 0$ . Comme les dérivées d'une fonction harmonique à l'intérieur du domaine sont contrôlées par l'oscillation de la fonction sur le domaine, il est attendu que la même propriété soit conservée pour des petites perturbations du laplacien dans des espaces fonctionnels appropriés. Plus précisément, considérons le cas particulier d'une équation linéaire uniformément elliptique sous la forme non-divergence

$$\sum_{1 \leq i, j \leq d} a_{ij}(x) \partial_{ij} u = g(x).$$

Trois classes générales d'estimations peuvent être étudiées :

- *Estimations Cordes-Nirenberg*. Soit  $0 < \alpha < 1$  et supposons que  $\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \delta = \delta(\alpha)$  avec  $\delta$  petit. Alors

$$u \in C^{1, \alpha}(\overline{B}_{1/2}) \quad \text{et} \quad \|u\|_{C^{1, \alpha}(\overline{B}_{1/2})} \leq C (\|u\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)}).$$

- *Estimations Schauder*. Si  $g$  et les coefficients  $a_{ij}$  sont dans  $C^\alpha(\overline{B}_1)$ , alors

$$u \in C^{2, \alpha}(\overline{B}_{1/2}) \quad \text{et} \quad \|u\|_{W^{2, p}(B_{1/2})} \leq C (\|u\|_{L^\infty(B_1)} + \|g\|_{L^p(B_1)}).$$

- *Estimations Calderón-Zygmund.* Si les coefficients  $a_{ij}$  sont continues sur  $B_1$  et  $g \in L^p(B_1)$ , pour un certain  $1 < p < +\infty$ , alors

$$u \in W^{2,p}(B_{1/2}) \quad \text{et} \quad \|u\|_{W^{2,p}(B_{1/2})} \leq C (\|u\|_{L^\infty(B_1)} + \|g\|_{L^p(B_1)}).$$

La principale stratégie pour obtenir ces estimées utilise une estimation d'Aleksandrov-Bakelman-Pucci (ABP), le principe du maximum et un argument de recouvrement afin de :

- Contrôler la fonction de répartition d'une solution ; cela conduit à l'inégalité de Harnack, et par conséquent à la régularité  $C^\alpha$ .
- Approcher en norme  $L^\infty$  une solution par des fonctions affines (resp. des paraboloides) ; cela mène aux estimations  $C^{1,\alpha}$  (resp.  $C^{2,\alpha}$ ).
- Contrôler la courbure des paraboloides touchant la solution par-dessus et par-dessous ; cela aboutit aux estimations  $W^{2,\varepsilon}$  et  $W^{3,\varepsilon}$ .

Par conséquent, un point-clé est de comprendre et de comparer les dérivées partielles d'une fonction par rapport à ses approximations polynomiales. La méthode globale décrite ci-dessus est fondamentalement non-linéaire, au sens où elle ne repose pas sur la différentiation de l'équation (5). Par conséquent, les résultats ainsi obtenus peuvent s'appliquer à des équations non-linéaires générales comme les équations de Pucci, de Bellman et d'Isaacs (voir exemples 1–2 de la section 0.1.1). Pour une étude de la régularité par la méthode linéaire de différentiation de l'équation (5), nous renvoyons aux travaux de Caffarelli, Kohn, Nirenberg et Spruck [30, 29], Evans [60] et Krylov [92].

Les *inégalités de Harnack* expriment le fait que le maximum dans un ouvert d'une solution d'une EDP elliptique est contrôlé par l'infimum de cette solution dans l'ouvert. En 1979, Krylov et Safonov [96, 97] ont prouvé l'inégalité de Harnack pour les solutions du second ordre d'équations elliptiques sous forme non-divergence avec des coefficients mesurables. Ils ont ouvert la voie au développement d'une théorie de la régularité pour les équations non-linéaires qui, en particulier, ne s'écrivent pas sous forme divergence. Pour les solutions de viscosité de (5) supposée uniformément elliptique avec  $g$  continue bornée et  $u \geq 0$ , l'inégalité de Harnack énonce qu'il existe une constante universelle  $C > 0$  (ne dépendant que de la dimension et des constantes d'ellipticité) telle que

$$\sup_{B_{1/2}} u \leq C \left( \inf_{B_{1/2}} u + \|g\|_{L^d(B_1)} \right).$$

La conséquence principale - et très classique - de l'inégalité de Harnack est la régularité  $C^\alpha$  des solutions de (5). La preuve de l'inégalité de Harnack repose sur une inégalité ABP établie par Caffarelli pour les solutions de viscosité, combinée aux arguments de Krylov et Safonov (voir aussi la partie 9.8 de la monographie de Gilbarg et Trudinger [74]). Avec le principe du maximum, l'inégalité ABP est l'un des principaux outils utilisés au coeur de la théorie de la régularité pour les solutions de viscosité. Parmi les nombreuses versions existantes, nous avons choisi de présenter ici un corollaire de la preuve de [34, théorème 3.2] qui est énoncé dans [5, proposition 3.2].

**Proposition 2.** (*Inégalité ABP elliptique*) Soit  $g \in C(B_1)$ . Si  $u \in \text{LSC}(\overline{B_1})$  satisfait

$$\begin{cases} \mathcal{P}_{\lambda,\Lambda}^+(D^2u) \geq -g & \text{sur } B_1, \\ u \geq 0 & \text{sur } \partial B_1, \end{cases}$$

alors  $u_- := -\min\{0, u\}$  vérifie

$$u_-(0) \leq \lambda^{-1} \left( \frac{1}{|B_1|} \int_{\{\Gamma_u=u\}} g_+^d(x) dx \right)^{1/d}$$

où  $\Gamma_u(x) := \sup_{p \in \mathbb{R}^d} \inf_{y \in B_1} (-u_-(y) + p \cdot (x - y))$  est l'enveloppe convexe de  $-u_-$ .

Nous allons maintenant aborder la procédure de régularisation par *inf-sup convolution*. Cette méthode, très utile pour l'étude des solutions de viscosité, a été introduite par Lasry et Lions [100] dans le cadre général des espaces de Hilbert. Dans le cas elliptique, pour  $\delta > 0$ , l'inf-convolution  $u_\delta^-$  et la sup-convolution  $u_\delta^+$  d'une fonction continue  $u : U \rightarrow \mathbb{R}$  s'écrivent

$$u_\delta^-(x) := \inf_{y \in A} \left( u(y) + \frac{1}{\delta} |x - y|^2 \right) \quad \text{et} \quad u_\delta^+(x) := \sup_{y \in A} \left( u(y) - \frac{1}{\delta} |x - y|^2 \right).$$

où  $A$  est l'ensemble sur lequel est effectuée la régularisation. Elle a été utilisée de façon cruciale par Jensen [86] pour obtenir l'unicité des solutions de viscosité pour le problème de Dirichlet pour les équations du second ordre. En utilisant cette procédure de régularisation, Caffarelli [31] a montré :

**Proposition 3** (Estimation  $C^{1,\alpha}$  intérieure). *Si  $\mathcal{F}$  est uniformément elliptique, alors la solution  $u$  de (5) est  $C^{1,\alpha}(\overline{B}_{1/2})$  pour un certain  $0 < \alpha < 1$  et il existe une constante universelle  $C$  telle que*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|g\|_{L^d(B_1)} \right).$$

La preuve de Jensen a permis l'élaboration d'une théorie de la régularité en montrant que la régularité des solutions est préservée sous certaines perturbations des équations (voir [27, 33]). Des résultats connexes furent obtenus par d'autres moyens par Safonov [123, 124] et Trudinger [140, 141].

Pour un opérateur  $\mathcal{F}$  concave, Evans [60] par la méthode de la continuité et Krylov [90, 91] ont montré que les solutions de viscosité sont  $C^{2,\alpha}$  et satisfont une estimation intérieure  $C^{2,\alpha}$  a priori dépendant de  $\|u\|_{L^\infty}$  (voir par exemple [74] pour l'élaboration de la méthode de continuité).

La solution de viscosité  $u$  de (5) est dite *classique* de (5) si  $u \in C^2(\Omega)$  et  $u$  satisfait (5). Une question naturelle est alors d'étudier le degré de régularité, au moins à partir de la classe  $C^2$  (comme  $C^{2,\alpha}$  dans le cas du théorème d'Evans-Krylov) pour d'autres classes d'opérateurs  $\mathcal{F}$  éventuellement plus générales et de déterminer de telles classes. Cette question est restée longtemps ouverte. Nadirashvili et Vladut se sont intéressés à l'existence de solutions non-classiques en exhibant des contre-exemples en grande dimension  $d \geq 12$  [113, 114] puis ont réussi à faire diminuer la dimension à  $d = 5$  [116], puis  $d = 4$  [115]. Ils montrent que les dérivées secondes des solutions de viscosité d'une équation  $\mathcal{F}(D^2u) = 0$  uniformément elliptique peuvent exploser sur un voisinage d'un point et donc sur un ensemble singulier de manière constructiviste en considérant explicitement des fonctions de la forme :  $w_d(x) = P_d(x)/|x|^\delta$  où  $\delta \in [1, 2]$ ,  $d \in \{4, 5, 12\}$  est la dimension ambiante et  $P_d$  est une forme cubique (minimale). Ils prouvent qu'une régularité intérieure optimale de telles solutions ne saurait être espérée au-delà de  $C^{1+\epsilon}$ , montrant par conséquent l'optimalité de la régularité  $C^{1,\alpha}$  dans le cas général. De plus, ils établissent dans [115] que  $d = 5$  est la dimension minimale pour laquelle il existe des solutions homogènes d'ordre 2 non-régulières pour des équations elliptiques non-linéaires uniformément elliptiques.

Cependant, les solutions de viscosité sont  $C^{2,\epsilon}$ -régulières presque partout. En fait, c'est vrai sur l'ensemble complémentaire d'un ensemble fermé de dimension de Hausdorff strictement plus petite que  $d$ . Le cas elliptique a été résolu par Armstrong, Smart et Silvestre [2]. Ils se sont intéressés à :

$$\mathcal{F}(D^2u) = 0 \quad \text{sur } U \subseteq \mathbb{R}^d. \quad (6)$$

L'énoncé précis du résultat est le suivant :

**Théorème 4** (Régularité partielle elliptique, Armstrong, Smart et Silvestre). *Supposons que l'opérateur  $\mathcal{F}$  satisfait (F1) et (F2). Soit  $u \in C(U)$  une solution de viscosité de (6) sur un domaine  $U \subseteq \mathbb{R}^d$ . Alors il existe une constante  $\varepsilon > 0$ , dépendant seulement de  $d, \lambda, \Lambda$  et un sous-ensemble fermé  $\Sigma \subseteq \overline{U}$  de dimension de Hausdorff au plus  $d - \varepsilon$ , tel que, pour tout  $0 < \alpha < 1$ , la solution  $u$  est de classe  $C^{2,\alpha}(U \setminus \Sigma)$ .*

Ils obtiennent ce résultat en combinant des estimées  $W^{2,\varepsilon}$  elliptiques avec un lemme de régularité établi par Savin [127]. La notion d'estimée  $W^{2,\varepsilon}$  remonte à un résultat initialement obtenu par Lin [105] concernant une estimation intégrale portant sur les dérivées secondes de solutions fortes d'équations

linéaires, uniformément elliptiques sous la forme non-divergence et possédant des coefficients mesurables (voir par exemple [28] pour ce cadre). Ce résultat a ensuite été étendu au cadre des solutions de viscosité non-linéaires uniformément elliptiques par Caffarelli [31] (voir aussi [34]).

Nous allons maintenant expliquer l'estimée  $W^{3,\varepsilon}$  elliptique. Pour cela, nous introduisons la quantité

$$\Psi(u, U)(x) := \inf \left\{ A \geq 0 : \exists (p, M) \in \mathbb{R}^d \times \mathbb{M}_d \text{ tels que } \forall y \in U, \right. \\ \left. |u(y) - u(x) - p \cdot (y - x) - \frac{1}{2}(y - x) \cdot M(y - x)| \leq \frac{1}{6}A|x - y|^3 \right\}.$$

La quantité  $\Psi(u, U)(x)$  est définie en termes de polynômes cubiques (ordre 3 en espace) qui touchent  $u$  au point  $x$  et qui sont respectivement au-dessus ou en-dessous de  $u$  sur le domaine  $U$  tout entier.

**Théorème 5** (Estimation  $W^{3,\varepsilon}$  elliptique, Armstrong, Smart et Silvestre). *Supposons que l'opérateur  $\mathcal{F}$  satisfait (F1) et  $u \in C(B_1)$  est une solution de (6) sur  $B_1$ . Alors il existe deux constantes universelles  $C, \varepsilon > 0$  telles que, pour tout  $\kappa > 0$ ,*

$$|\{(x, t) \in B_{1/2} : \Psi(u, B_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{B_1} |u| + |\mathcal{F}(0)|} \right)^{-\varepsilon}.$$

### Quelques aspects de régularité parabolique :

Une première introduction aux solutions de viscosité aux équations non-linéaires paraboliques peut être trouvée dans [78]. Nous indiquons aussi la série des trois papiers de L. Wang[143, 144, 145] pour avoir un premier aperçu pour l'étude de la régularité pour les solutions de viscosité dans le cas parabolique. Pour une théorie  $L^p$ , nous renvoyons à [46]. Nous renvoyons pour une théorie moderne des équations non-linéaires paraboliques à [92, 104]. Puisque nous l'utiliserons dans la suite, nous donnons la proposition suivante.

**Proposition 6** (Estimation  $C^{1,\alpha}$  intérieure parabolique [144, Section 4.2]). *Soit  $u$  une solution de viscosité de*

$$\partial_t u + \mathcal{F}(D^2 u) = g \quad \text{sur } Q_1,$$

*Alors  $u \in C^{1,\alpha}(\overline{Q}_{1/2})$  pour une certaine constante universelle  $0 < \alpha < 1$ . De plus, il existe une constante universelle  $C$  telle que*

$$\sup_{\overline{Q}_{1/2}} |Du| \leq C \left( \sup_{Q_1} |u| + |\mathcal{F}(0)| + \|g\|_{C^{0,1}(Q_1)} \right).$$

Les régularisations par sup et inf-convolutions sont analogues au cas elliptique et sont données respectivement par :

$$u_\delta^+(x, t) = \sup_{(y,s) \in H} \left( u(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s)^2) \right)$$

et

$$u_\delta^-(x, t) = \inf_{(y,s) \in H} \left( u(y, s) + \frac{1}{2\delta} (|x - y|^2 + (t - s)^2) \right).$$

Pour plus de détails, nous renvoyons le lecteur à la section 8 du User's guide [45].

## 2 Les résultats de la thèse

### 2.1 Le problème de Neumann pour les équations elliptiques et paraboliques non-linéaires

Cette section reprend les résultats du chapitre 1 et de l'article [50]. Nous nous intéressons ici à des équations elliptiques non-linéaires elliptiques et paraboliques de la forme

$$-\partial_t u + \mathcal{F}(t, x, u, Du, D^2 u) = 0 \quad \text{sur } \Omega \subseteq \mathbb{R}^d \times (-\infty, T), \quad (7)$$

ou

$$\mathcal{F}(x, u, Du, D^2u) + \lambda u = 0 \quad \text{sur } U \subseteq \mathbb{R}^d, \quad (8)$$

où  $\mathcal{F}$  satisfait la condition d'ellipticité (E), munies d'une condition au bord inhomogène de Neumann

$$\frac{\partial u}{\partial n} = h \quad \text{sur } \partial\Omega,$$

où  $n(x)$  désigne le vecteur normal unitaire au bord  $\partial\Omega$  en  $x$  dirigée vers l'extérieur de  $\Omega$ . Kohn et Serfaty [88] ont étudié le problème de Dirichlet associé à (7) et (8) (voir section 0.1.3.2). Notre motivation est ici d'adapter leur approche au problème de Neumann dans les deux cadres elliptique et parabolique. Puis, une fois que ce problème sera résolu, nous pourrions alors expliquer de façon naturelle l'extension à des conditions de bord de type oblique ou mixte Dirichlet-Neumann.

### 2.1.1 Discussion sur le choix de la dynamique, hypothèses sur le domaine et notations

**2.1.1.1 Le choix de la dynamique** Suivant la discussion faite, le type de condition au bord influe sur la dynamique du processus choisi.

La table 2 rassemble les interprétations en termes de dynamiques, stochastiques ou déterministes, pour certaines conditions aux limites classiques.

Condition au bord	Dynamique
Dirichlet $g$	temps d'arrêt + bonus $g$
Neumann $h$	réflexion normale sur le bord + bonus $h$
Oblique $h$	réflexion oblique sur le bord + bonus $h$

Tableau 2 – Conditions aux limites et dynamiques associées. La condition oblique inhomogène s'écrit  $\frac{\partial u}{\partial \gamma} = h$  où  $\gamma$  est un champ de vecteurs sur  $\partial\Omega$  satisfaisant  $\gamma(x) \cdot n(x) \geq \nu > 0$  sur  $\partial\Omega$ , et  $h \in C(\partial\Omega)$ .

Pour le jeu de Paul et Carol associé à l'équation du mouvement par courbure moyenne (rappelé à la section 0.1.3.1), le problème de Neumann avec une condition homogène  $h = 0$  a été étudié par Giga et Liu [73]. En partant de l'idée naturelle qu'une condition de Neumann homogène se modélise par une dynamique de réflexion normale sur le bord (voir Table 2), ils ont introduit des règles de billard dans le jeu afin d'étendre l'interprétation faite par Kohn et Serfaty [87]. Mais dans le cas du jeu de Mark et Helen, si l'on veut donner une interprétation par billard, il s'est avéré que nous avons seulement pu traiter le cas homogène.

Une autre approche consiste à pénaliser la dynamique du processus. Pour un domaine convexe borné, Lions, Menaldi et Snitzman [107] construisent une séquence d'équations différentielles stochastiques avec un terme de drift qui pénalise fortement le processus s'il souhaite quitter le domaine. La limite de la suite des processus stochastiques associés converge vers un processus de diffusion qui se réfléchit suivant le vecteur normal sur la frontière. Barles et Lions [17] ont traité le cas oblique en étudiant précisément les liens entre des processus approchés et les opérateurs elliptiques associés aux dynamiques stochastiques obliques originales.

Au lieu du billard, nous avons procédé par une pénalisation sur la dynamique de la particule, qui dépend de la condition de Neumann  $h$  au bord. Elle est favorable à l'un ou l'autre des joueurs suivant son signe. Nous modifions les règles du jeu habituelles données dans [88] dans un voisinage du bord suffisamment petit. La particule, dont le mouvement est contrôlé par l'un des joueurs, peut quitter le domaine mais si c'est le cas, nous la ramenons sur le bord du domaine par une projection orthogonale sur le bord (qui sera définie de façon légitime). Ce mouvement de la particule, combiné à un poids adapté associé à la condition de Neumann, donne la pénalisation convenable. En dehors de cette région qui délimite ce voisinage du bord, les règles habituelles sont conservées.



**2.1.1.2 Hypothèses sur la régularité du domaine et la non-linéarité** Après avoir donné quelques notations préliminaires, nous donnerons les hypothèses considérées portant sur la régularité du domaine ainsi que sur la non-linéarité.

**Notations :** Si  $\Omega$  est un domaine régulier, disons de classe  $C^2$ , la fonction distance au bord  $\partial\Omega$  est notée  $\text{dist} = \text{dist}(\cdot, \partial\Omega)$ , et nous rappelons que, pour tout  $x \in \partial\Omega$ , la normale extérieure au bord  $n(x)$  au bord  $\partial\Omega$  au point  $x$  est donné par  $n(x) = -D \text{dist}(x)$ . Observons que, si le domaine est supposé borné et au moins de classe  $C^2$ , tout  $x \in \mathbb{R}^d$  étant dans un voisinage suffisamment petit de la frontière admet une unique projection sur  $\partial\Omega$ , notée par

$$\bar{x} := \text{proj}_{\partial\Omega}(x). \quad (9)$$

En particulier, le vecteur  $x - \bar{x}$  est parallèle à  $n(\bar{x})$ . Lorsqu'elle est bien définie, la projection sur  $\bar{\Omega}$  est notée par  $\text{proj}_{\bar{\Omega}}$  et peut être décomposée par

$$\text{proj}_{\bar{\Omega}}(x) = \begin{cases} \text{proj}_{\partial\Omega}(x), & \text{si } x \notin \Omega, \\ x, & \text{si } x \in \Omega. \end{cases}$$

Nous rappelons également les conditions géométriques classiques suivantes (voir e.g. [61]).

**Définition 1** (Condition de boule intérieure/extérieure uniforme). *Le domaine  $\Omega$  satisfait la condition de boule intérieure en  $x_0 \in \partial\Omega$  s'il existe une boule ouverte  $B \subseteq \Omega$  avec  $x_0 \in \partial B$ .*

*Le domaine  $\Omega$  satisfait la condition de boule intérieure uniforme s'il existe  $r > 0$  tel que pour tout  $x \in \partial\Omega$  il existe une boule ouverte  $B \subseteq \Omega$  avec  $x \in \partial B$  et rayon  $r$ . De plus, le domaine  $\Omega$  satisfait la condition de boule extérieure uniforme si  $\mathbb{R}^d \setminus \bar{\Omega}$  satisfait la condition de boule intérieure uniforme.*

Nous pouvons maintenant préciser les hypothèses sur le domaine  $\Omega$ . Nous supposons que le domaine  $\Omega$  – non nécessairement borné – satisfait :

(H1)  $\Omega$  est un domaine de classe  $C^2$  ;

(H2)  $\Omega$  vérifie des conditions de boule intérieure et extérieure uniformes.

On observe que la condition de boule intérieure uniforme implique la condition de boule intérieure et que les deux conditions de boule intérieure / extérieure sont vérifiées automatiquement pour un domaine borné de classe  $C^2$ . Ces conditions permettent de définir correctement la notion de projection dans un voisinage tubulaire du bord et d'éviter – si le domaine n'est pas borné – certaines singularités géométriques pour le comportement du jeu, notamment que la particule saute entre des parties du domaine éloignées pour la distance géodésique.

On termine cette présentation en listant nos hypothèses principales sur la forme de la non-linéarité. La fonction  $\mathcal{F}$  à valeurs réelles dans (7) est définie sur  $\mathbb{R} \times \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$ . Elle est supposée être une fonction continue de toutes ses variables, et être elliptique en satisfaisant la condition d'**ellipticité** (E). Dans le cadre dépendant aussi du temps (7), nous permettons une croissance linéaire à  $\mathcal{F}$  en  $|z|$  (de telle sorte que les solutions peuvent croître exponentiellement, mais ne pas exploser). Cependant, nous demandons un contrôle uniforme en  $x$  (de sorte que les solutions restent bornées lorsque  $\|x\| \rightarrow \infty$  pour  $t$  fixé). En fait, nous supposons que :

- $\mathcal{F}$  a une croissance au plus linéaire en  $z$  au voisinage du point  $(p, \Gamma) = 0$  au sens où pour tout  $K$  nous avons

$$|\mathcal{F}(t, x, z, p, \Gamma)| \leq C_K(1 + |z|), \quad (10)$$

pour une certaine constante  $C_K \geq 0$ , pour tout  $x \in \bar{\Omega}$  et  $t, z \in \mathbb{R}$ , lorsque  $\|(p, \Gamma)\| \leq K$ .

- $\mathcal{F}$  est localement Lipschitz en  $p$  et  $\Gamma$  au sens où pour tout  $K$  nous avons

$$|\mathcal{F}(t, x, z, p, \Gamma) - \mathcal{F}(t, x, z, p', \Gamma')| \leq C_K(1 + |z|)\|(p, \Gamma) - (p', \Gamma')\|, \quad (11)$$

pour une certaine constante  $C_K \geq 0$ , pour tout  $x \in \bar{\Omega}$  et  $t, z \in \mathbb{R}$ , lorsque  $\|(p, \Gamma)\| + \|(p', \Gamma')\| \leq K$ .

- $\mathcal{F}$  a une croissance contrôlée par rapport à  $p$  et  $\Gamma$ , au sens que, pour certaines constantes  $q, r \geq 1$ ,  $C > 0$ , nous avons

$$|\mathcal{F}(t, x, z, p, \Gamma)| \leq C(1 + |z| + \|p\|^q + \|\Gamma\|^r), \quad (12)$$

pour tout  $t, x, z, p$  et  $\Gamma$ .

Dans le cas stationnaire (8) nos solutions seront uniformément bornées. Pour prouver l'existence de telles solutions, le facteur d'actualisation devrait être suffisamment grand. Nous aurons aussi besoin d'hypothèses analogues à celles du cas parabolique mais ils peuvent être locales en  $z$  car  $z$  sera finalement à valeurs dans un ensemble compact. En fait, nous supposons que :

- Il existe  $\eta > 0$  tel que pour tout  $K \geq 0$ , il existe  $C_K^* > 0$  satisfaisant

$$|\mathcal{F}(x, z, p, \Gamma)| \leq (\lambda - \eta)|z| + C_K^*, \quad (13)$$

pour tout  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$ , tel que  $\|(p, \Gamma)\| \leq K$ ; ici  $\lambda$  est le coefficient qui apparaît devant  $u$  dans l'équation (8).

- $\mathcal{F}$  est localement Lipschitz en  $p$  et  $\Gamma$  au sens où nous avons

$$|\mathcal{F}(x, z, p, \Gamma) - \mathcal{F}(x, z, p', \Gamma')| \leq C_{K,L} \|(p, \Gamma) - (p', \Gamma')\|, \quad (14)$$

pour une certaine constante  $C_{K,L} \geq 0$ , pour tout  $x \in \overline{\Omega}$ , lorsque  $\|(p, \Gamma)\| + \|(p', \Gamma')\| \leq K$  et  $|z| \leq L$ .

- $\mathcal{F}$  a une croissance contrôlée par rapport à  $p$  et  $\Gamma$ , au sens où pour certaines constantes  $q, r \geq 1$  et pour tout  $L$  nous avons

$$|\mathcal{F}(x, z, p, \Gamma)| \leq C_L(1 + \|p\|^q + \|\Gamma\|^r), \quad (15)$$

pour une certaine constante  $C_L \geq 0$ , pour tout  $x, p$  et  $\Gamma$ , et pour tout  $|z| \leq L$ .

Pour la suite de la présente introduction, nous avons choisi d'expliquer soigneusement le problème parabolique dans le cas particulier :

$$\boxed{\mathcal{F} \text{ ne dépend pas de la variable } u}. \quad (16)$$

Ce choix a été effectué par souci de clarté pour la présentation, l'étude complète et rigoureuse du cas général étant faite au chapitre 2. En particulier, il en découle les conséquences suivantes :

- ◇ L'EDP possède automatiquement un principe de comparaison.
- ◇ Le principe de programmation dynamique donne une équation de récurrence explicite pour la fonction score  $u^\varepsilon$ . Dans le cas où  $\mathcal{F}$  dépend de  $u$ , il nous a été nécessaire d'utiliser une formulation par lignes de niveaux et le principe de programmation dynamique donne une équation sur une quantité  $U^\varepsilon$ , qui a une forme similaire à (4). On déduit alors deux inéquations de programmations dynamiques sur deux fonctions scores  $u^\varepsilon$  et  $v^\varepsilon$ . De plus, ces relations sont alors implicites en temps.

Une fois que l'on aura traité le cas parabolique, nous présenterons le cas elliptique général (voir section 0.2.1.3) ainsi que des généralisations naturelles et des perspectives possibles tels que le problème elliptique mixte Dirichlet-Neumann et le problème parabolique pour un champ de vecteurs oblique (voir section 0.2.1.4).

### 2.1.2 Le cas parabolique sans la dépendance en $u$

On se donne le problème parabolique général avec condition finale  $g$  et condition de Neumann  $h$  de la forme

$$\begin{cases} -\partial_t u + \mathcal{F}(t, x, Du, D^2u) = 0, & \text{pour } x \in \Omega \text{ et } t < T, \\ \langle Du(x, t), n(x) \rangle = h(x), & \text{pour } x \in \partial\Omega \text{ et } t < T, \\ u(x, T) = g(x), & \text{pour } x \in \overline{\Omega}, \end{cases} \quad (17)$$

où

- $\mathcal{F}(t, x, p, \Gamma)$  est continue dans toutes ses variables, satisfait la condition (E) de monotonie dans sa dernière variable et vérifie les propriétés (10)–(12) ;
- $\Omega$  est un domaine de bord  $C^2$  satisfaisant les conditions de boule intérieure et extérieure ;
- $g$  et  $h$  sont continues, uniformément bornées et dépendent seulement de la position  $x$ .

Les EDP que l'on considère peuvent être dégénérées paraboliques, dégénérées elliptiques, ou même du premier ordre. Par conséquent, nous ne pouvons pas nous attendre à une solution classique, et il n'est pas toujours possible de comprendre les conditions au bord au sens classique (il est cependant possible de le faire si nous imposons une condition d'uniforme ellipticité comme il est montré dans le chapitre 3). La théorie des solutions de viscosité fournit un cadre adapté et pratique pour manipuler ces problèmes. Nous allons dès maintenant procéder un passage en revue des définitions de base que nous aurons besoin dans notre cadre par souci de lisibilité pour le lecteur. Nous renvoyons à l'ouvrage de Barles [11], au "User's guide" de Crandall, Ishii et Lions [45] et aux livres de Bardi et Capuzzo Dolcetta [8] et de Giga [72] pour davantage de détails au sujet de la théorie générale. Le lecteur est renvoyé plus spécifiquement à [45, 10, 79] pour la théorie générale des solutions de viscosité pour les équations non-linéaires avec condition de Neumann. Pour la condition au bord de Neumann, sa relaxation au sens de viscosité a été initialement proposé par Lions [106].

Les résultats obtenus dans cette partie sont surtout intéressants si l'EDP possède un *principe de comparaison*, c'est-à-dire que toute sous-solution est en-dessous de toute sur-solution. Pour de telles équations, nous déduisons l'existence de la limite de  $u^\varepsilon$  et que cette limite est l'unique solution de viscosité. Dans le cas où  $\mathcal{F}$  est continue en chacune de ses variables, de nombreux principes de comparaison existent et de résultats d'existence pour les solutions de viscosité d'EDP paraboliques du second ordre avec des conditions générales de type Neumann. Nous référons à [10, 79, 12, 106]. Pour des conditions de Neumann homogènes, Sato a obtenu un principe de comparaison pour certaines EDP paraboliques.

Nous allons expliquer en quel sens nous devons comprendre une solution de viscosité pour (17). Nous devons faire attention à imposer la condition au bord au sens de viscosité.

**Définition 2.** Une fonction réelle semi-continue inférieurement  $u(x, t)$  définie pour  $x \in \overline{\Omega}$  et  $t_* \leq t \leq T$  est une sur-solution de viscosité du problème à temps final (17) si

(P1) pour tout  $(x_0, t_0)$  avec  $x_0 \in \Omega$  et  $t_* \leq t_0 < T$  et toute fonction  $\phi(x, t)$  régulière telle que  $u - \phi$  a un minimum local en  $(x_0, t_0)$ , nous avons

$$-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0,$$

(P2) pour tout  $(x_0, t_0)$  avec  $x_0 \in \partial\Omega$  et  $t_* \leq t_0 < T$  et toute fonction  $\phi(x, t)$  régulière telle que  $u - \phi$  a un minimum local en  $\overline{\Omega}$  en  $(x_0, t_0)$ , nous avons

$$\max\{-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)), \langle D\phi(x_0, t_0), n(x_0) \rangle - h(x_0)\} \geq 0,$$

(P3)  $u \geq g$  à l'instant final  $t = T$ .

De façon similaire, une fonction réelle semi-continue supérieurement  $u(x, t)$  définie pour  $x \in \overline{\Omega}$  et  $t_* \leq t \leq T$  est une sous-solution de viscosité du problème à l'instant final (17) si

(P1) pour tout  $(x_0, t_0)$  avec  $x_0 \in \Omega$  et  $t_* \leq t_0 < T$  et toute fonction  $\phi(x, t)$  régulière telle que  $u - \phi$  a un maximum local en  $(x_0, t_0)$ , nous avons

$$-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0,$$

(P2) pour tout  $(x_0, t_0)$  avec  $x_0 \in \partial\Omega$  et  $t_* \leq t_0 < T$  et toute fonction  $\phi(x, t)$  telle que  $u - \phi$  a un maximum local sur  $\overline{\Omega}$  en  $(x_0, t_0)$ , nous avons

$$\min\{-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)), \langle D\phi(x_0, t_0), n(x_0) \rangle - h(x_0)\} \leq 0,$$

(P3)  $u \leq g$  à l'instant final  $t = T$ .

Une solution de viscosité de (17) est une fonction continue  $u$  qui est à la fois une sous-solution de viscosité et une sur-solution de viscosité de (17).

Pour obtenir le résultat désiré de la représentation de la solution de l'EDP au sens de viscosité par le jeu, la principale question est de définir correctement le jeu au voisinage du bord, d'une part la dynamique de la particule et d'autre part comment la présence de la particule au voisinage du bord affecte l'évolution du score d'Helen.

Il y a deux joueurs Mark et Helen et nous fixons un petit paramètre  $\varepsilon$ . L'état du jeu est décrit par sa position spatiale  $x \in \overline{\Omega}$  et le score  $y \in \mathbb{R}$  d'Helen. L'objectif d'Helen est de maximiser son score à l'instant final, tandis que le but de Mark est de s'opposer à elle, et donc son but est au contraire de minimiser son score. Puisque l'EDP est résolue sur  $\Omega$ , la condition finale  $g$  va être interprétée dans le jeu comme un bonus à la fin de la partie pour Helen à l'instant final tandis que la condition de Neumann  $h$  correspond dans une interprétation financière à un coupon que touche Helen si le déplacement fait sortir la particule du domaine (avant d'être ramenée).

Les règles du jeu dépendent de trois nouveaux paramètres,  $\alpha, \beta, \gamma > 0$ , dont la présence ne génère aucune perte de généralité, choisis suffisamment petits et vérifiant certaines conditions. Certaines conditions sont liées à la régularité de  $\mathcal{F}$  (par les coefficients  $p, q$  et  $r$ ) soit pour que le jeu soit bien défini soit pour obtenir des résultats de convergence et de consistance. Les relations que ces paramètres doivent vérifier sont énoncées de façon précise à la sous-section 1.2.2 du chapitre 1. Pour la suite de cette section de l'introduction, nous noterons cependant que nous imposons – parmi d'autres conditions détaillées au chapitre 1 – les conditions

$$\alpha < 1/3 \quad (18)$$

et

$$\gamma < 1 - \alpha, \quad 2\alpha + \gamma < 2, \quad \gamma(r - 1) < 2\alpha. \quad (19)$$

Ces conditions ne restreignent pas la classe d'équations qu'on considère, car pour tout  $q$  et  $r$  il existe  $\alpha, \beta$  and  $\gamma$  avec les propriétés désirées.

Lorsque le jeu commence, à l'instant  $t_0$ , la position spatiale peut prendre n'importe quelle valeur  $x_0 \in \mathbb{R}^d$  et le score d'Helen est  $y_0 = 0$ . Les règles sont les suivantes : si à l'instant  $t_j = t_0 + j\varepsilon^2$ , la position est  $x_j$  et le score d'Helen est  $y_j$ , alors

**Etape 1 :** Helen choisit un vecteur  $p_j \in \mathbb{R}^d$  et une matrice  $\Gamma_j \in \mathcal{S}_d$ , restreints par

$$\|p_j\| \leq \varepsilon^{-\beta} \quad \text{et} \quad \|\Gamma_j\| \leq \varepsilon^{-\gamma}. \quad (20)$$

**Etape 2 :** Mark voit le choix d'Helen. En le prenant en compte, Mark choisit une position  $x_{j+1}$  de sorte à diminuer le score d'Helen. Mark choisit un point intermédiaire  $\hat{x}_{j+1} = x_j + \Delta\hat{x}_j \in \mathbb{R}^d$  tel que

$$\|\Delta\hat{x}_j\| \leq \varepsilon^{1-\alpha}. \quad (21)$$

Ce point  $\hat{x}_{j+1}$  détermine la position  $x_{j+1} = x_j + \Delta x_j \in \overline{\Omega}$  à l'instant  $t_{j+1}$  suivant la règle

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}), \quad (22)$$

et le score d'Helen évolue selon la relation

$$y_{j+1} = y_j - \left( p_j \cdot \Delta\hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta\hat{x}_j, \Delta\hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(t_j, x_j, p_j, \Gamma_j) - \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j) \right).$$

**Etape 3 :** L'horloge avance à l'instant  $t_{j+1} = t_j + \varepsilon^2$  et le processus se répète, s'arrêtant lorsque  $t_K = T$ .

**Etape 4 :** A l'instant final, Helen reçoit  $g(x_K)$  provenant de l'option, où  $x_K$  est la position finale.

La fonction valeur d'Helen  $u^\varepsilon$  est déterminée par le principe de programmation dynamique

$$u^\varepsilon(x, t_j) = \max_{p, \Gamma} \min_{\Delta\hat{x}} [u^\varepsilon(x + \Delta x, t_{j+1}) - \left( p \cdot \Delta\hat{x} + \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, p, \Gamma) - \|\Delta\hat{x} - \Delta x\| h(x + \Delta x) \right)].$$

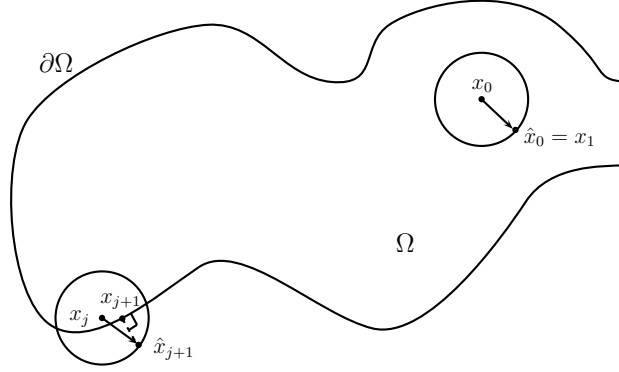


Figure 1 – Règles du jeu, mouvements admissibles près de la frontière et à l'intérieur du domaine.

couplé avec la condition finale  $u^\varepsilon(x, T) = g(x)$ . Ici  $\Delta\hat{x}$  est  $\hat{x}_{j+1} - x_j$  et  $\Delta x$  est déterminé par

$$\Delta x := x_{j+1} - x_j = \text{proj}_{\overline{\Omega}}(x_j + \Delta\hat{x}) - x_j. \quad (23)$$

Le max sur  $p, \Gamma$ , le minimum sur  $\Delta\hat{x}$  sont donnés par (20) et (21) et la définition de  $\Delta x$  est donnée par (23).

Pour notre résultat, nous allons concentrer notre attention sur les limites semi-relaxées inférieure et supérieure de  $u^\varepsilon$ , définies par (33) dans le cas dépendant du temps et (50) dans le cas stationnaire. Si  $u^\varepsilon(x, t)$  a une valeurs finie, nous montrons alors que la fonction  $u^\varepsilon(x, t)$  vérifie une équation de programmation dynamique qui peut être écrite sous la forme synthétique :

$$u^\varepsilon(x, t) = S_\varepsilon [x, t, u^\varepsilon(\cdot, t + \varepsilon^2)], \quad (24)$$

où  $S_\varepsilon [x, t, \phi]$  est défini pour  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$  et  $t \leq T$  et toute fonction continue  $\phi : \overline{\Omega} \rightarrow \mathbb{R}$  par

$$S_\varepsilon [x, t, \phi] := \max_{p, \Gamma} \min_{\Delta\hat{x}} [\phi(x + \Delta x) - \left( p \cdot \Delta\hat{x} + \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, p, \Gamma) - \|\Delta\hat{x} - \Delta x\| h(x + \Delta x) \right)], \quad (25)$$

soumise aux contraintes habituelles  $\|p\| \leq \varepsilon^{-\beta}$ ,  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ ,  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$  et

$$\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x}) - x.$$

L'opérateur  $S_\varepsilon$  satisfait les trois propriétés suivantes :

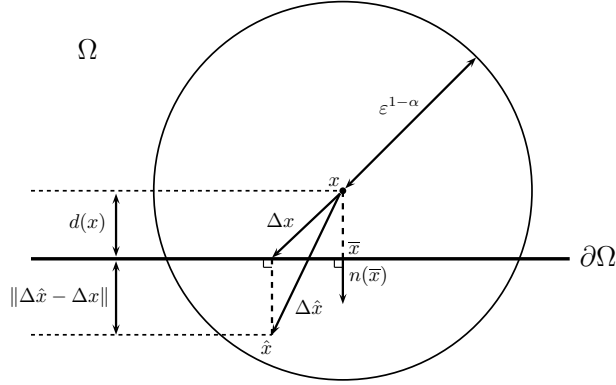
- ◇ Pour tout  $\phi \in C(\overline{\Omega})$ ,  $S_0 [x, t, \phi] = \phi(x)$ .
- ◇  $S_\varepsilon$  est *monotone*, i.e. pour tous  $\phi_1, \phi_2 \in C(\overline{\Omega})$ ,

$$\phi_1 \leq \phi_2 \implies S_\varepsilon [x, t, \phi_1] \leq S_\varepsilon [x, t, \phi_2]. \quad (26)$$

- ◇ Pour tout  $\phi \in C(\overline{\Omega})$  et  $c \in \mathbb{R}$ ,  $S_\varepsilon [x, t, c + \phi] = c + S_\varepsilon [x, t, \phi]$ .

**2.1.2.1 Dérivation formelle des stratégies optimales des joueurs** Nous expliquons ici formellement pourquoi  $u^\varepsilon$  devrait converger vers la solution de (17) lorsque  $\varepsilon$  tend vers 0. L'EDP (17) peut être considérée approximativement/grosso modo comme l'équation d'Hamilton-Jacobi-Bellman formelle associée au jeu à deux personnes présenté ci-dessus (voir la méthode décrite à la section 0.1.1).

La procédure pour la dérivation formelle à partir du principe de programmation dynamique vers l'EDP correspondante est classique : nous supposons que la fonction  $u^\varepsilon$  est régulière en effectuant un


 Figure 2 – Dérivation formelle pour  $x$  près du bord  $\partial\Omega$ , notation :  $\bar{x} = \text{proj}_{\partial\Omega}(x)$ .

développement de Taylor, puis nous supprimons la dépendance de  $u^\varepsilon$  vis-à-vis du paramètre  $\varepsilon$  et enfin nous faisons tendre  $\varepsilon$  vers 0. En exprimant la normale extérieure selon la relation

$$n(x + \Delta x) = -(\Delta x - \Delta \hat{x}) / \|\Delta \hat{x} - \Delta x\| \quad \text{si } \hat{x} \notin \Omega,$$

ce développement formel en supprimant la dépendance de  $u^\varepsilon$  sur  $\varepsilon$  conduit après réarrangement des termes, à :

$$0 \approx \varepsilon^2 \partial_t u + \max_{p, \Gamma} \min_{\Delta x} [(Du - p) \cdot \Delta \hat{x} + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - Du \cdot n(x + \Delta x)\} + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, p, \Gamma)]. \quad (27)$$

Nous allons distinguer deux cas suivant la valeur de  $\text{dist}(x, \partial\Omega)$ .

• **Cas 1 :**  $\text{dist}(x, \partial\Omega) > \varepsilon^{1-\alpha}$ . Tout d'abord, tout déplacement  $\Delta \hat{x}$  vérifie  $\Delta \hat{x} = \Delta x$  puisque  $B(x, \varepsilon^{1-\alpha}) \subseteq \Omega$  et cette expression se réduit simplement à

$$0 \approx \varepsilon^2 \partial_t u + \max_{p, \Gamma} \min_{\Delta x} \left[ (Du - p) \cdot \Delta x + \frac{1}{2} \langle (D^2 u - \Gamma) \Delta x, \Delta x \rangle - \varepsilon^2 \mathcal{F}(t, x, p, \Gamma) \right].$$

En considérant les problèmes d'optimisation aux premier et second ordres en  $\Delta x$ , Helen doit choisir :

$$p = Du \quad \text{et} \quad \Gamma \leq D^2 u.$$

En examinant le terme quadratique en  $\Delta x$ , le déplacement  $\Delta x = 0$  appartient toujours à l'ensemble des choix optimaux pour Mark et nous pouvons alors considérer que Mark choisit le déplacement  $\Delta x = 0$ . Il suffit alors à Helen de résoudre le problème d'optimisation  $\max\{-\varepsilon^2 \mathcal{F}(t, x, Du, \Gamma) : \Gamma \leq D^2 u\}$ , ce qui la conduit, par ellipticité (E) de  $\mathcal{F}$ , à choisir  $\Gamma = D^2 u$ .

• **Cas 2 :**  $\text{dist}(x, \partial\Omega) \leq \varepsilon^{1-\alpha}$ . On peut formellement assimiler la frontière à un hyperplan orthogonal au vecteur normal orienté vers l'extérieur  $n(\bar{x})$  où  $\bar{x}$  est la projection de  $x$  sur le bord  $\partial\Omega$  (voir (9) et Figure 1.2). Par le procédé de Gram-Schmidt, nous trouvons des vecteurs  $e_2, \dots, e_d$  tels que  $(e_1 = n(\bar{x}), e_2, \dots, e_d)$  forme une base orthonormale de  $\mathbb{R}^d$ . Dans cette base, on décompose le vecteur  $p$  en écrivant

$$p = (p \cdot n(\bar{x}))n(\bar{x}) + \tilde{p} \quad \text{avec} \quad \tilde{p} \cdot n(\bar{x}) = 0.$$

et la matrice  $\Gamma$  par  $\Gamma = (\langle \Gamma e_i, e_j \rangle)_{1 \leq i, j \leq d}$ . Nous allons nous concentrer sur le terme de pénalisation de la condition de Neumann notée par

$$P(x) = \|\Delta \hat{x} - \Delta x\| m(\Delta x) \quad \text{avec} \quad m(\Delta x) := [h(x + \Delta x) - Du(x) \cdot n(x + \Delta x)] \mathbb{1}_{\mathbb{R}^d \setminus \overline{\Omega}}(\hat{x}).$$

Si  $m(\Delta x) > 0$ , cette contribution est favorable à Helen car  $P(x) > 0$ , tandis que si  $m(\Delta x) < 0$  elle l'est à Mark car  $P(x) < 0$  et dans les deux cas son amplitude dépend de la norme du vecteur  $\Delta\hat{x} - \Delta x$ . Notre dérivation formelle est locale et essentiellement géométrique, au sens où notre objectif est de déterminer les choix optimaux pour Helen en considérant tous les déplacements  $\Delta\hat{x}$  possibles pour Mark. Par continuité de  $h$  et régularité de  $u$ , la fonction  $m(\Delta x)$  est proche de

$$m := [h(\bar{x}) - Du(x) \cdot n(\bar{x})] \mathbb{1}_{\mathbb{R}^d \setminus \overline{\Omega}}(\hat{x}). \quad (28)$$

Nous allons supposer ici que  $m(\Delta x)$  est localement constante sur la frontière et égale à  $m$  donnée par (1.2.30). Cette hypothèse correspond dans le jeu à supposer que dans un petit voisinage, traverser la frontière est toujours favorable à un joueur. Pour se concentrer sur les aspects géométriques du choix du déplacement de Mark, cette approche semble adaptée formellement puisqu'elle gèle les dépendances de  $P(x)$  vis-à-vis de  $m(x)$  en éliminant des difficultés liées aux variations locales de  $m(x)$  comme le changement de signe. Par conséquent, il suffit d'examiner

$$\max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (Du - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| m + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, p, \Gamma) \right]. \quad (29)$$

Pour déterminer les choix optimaux pour Helen du vecteur  $p$  et de la matrice  $\Gamma$ , on examine de façon conjointe les problèmes d'optimisation aux premier et second ordre dérivés cette fois de (29) et une analyse minutieuse détaillée au chapitre 1, sous-section 1.2.2.1 aboutit à :

$$p_{\text{opt}}(x) = Du + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m - \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2 u)_{11} \right] n(\bar{x}) \quad (30)$$

et

$$\Gamma_{\text{opt}}(x) = D^2 u + \left[ \frac{1}{2} \left( -1 + \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2 u)_{11} \right] E_{11}. \quad (31)$$

On revient au problème d'optimisation initial donné par (1.2.28). Si  $m = 0$ , en prenant la limite  $\varepsilon \rightarrow 0$ , on obtient  $h(x) - Du(x) \cdot n(x) = 0$ . Sinon, (1.2.28) se réduit formellement à

$$0 \approx \varepsilon^2 u_t + \begin{cases} \frac{1}{2} (\varepsilon^{1-\alpha} - \text{dist}(x)) m - \varepsilon^2 \mathcal{F}(t, x, p_{\text{opt}}(x), \Gamma_{\text{opt}}(x)) + o(\varepsilon^2), & \text{si } \text{dist}(x) \leq \varepsilon^{1-\alpha} \text{ et } m < 0, \\ -\varepsilon^2 \mathcal{F}(t, x, Du, D^2 u), & \text{si } \text{dist}(x) \geq \varepsilon^{1-\alpha} \text{ ou } m > 0, \end{cases} \quad (32)$$

avec  $(p_{\text{opt}}, \Gamma_{\text{opt}})$  donné par (1.2.48) et (1.2.52). Si  $x \in \Omega$ , pour  $\varepsilon$  assez petit, la seconde relation dans (1.2.53) est toujours valide de sorte qu'on déduit que  $u_t - \mathcal{F}(t, x, Du, D^2 u) = 0$ . Si  $x \in \partial\Omega$ ,  $\text{dist}(x) = 0$  et on distingue les cas  $m > 0$  et  $m < 0$ . Si  $m > 0$ , une fois de plus la seconde relation dans (1.2.53) est toujours valide de sorte que  $u_t - \mathcal{F}(t, x, Du, D^2 u) = 0$ . Sinon, si  $m < 0$ , la première relation dans (1.2.53) est toujours satisfaite. On observe que le terme du premier ordre en  $\varepsilon$  est prédominant puisque  $\varepsilon^{1-\alpha} \gg \varepsilon^2$ . En divisant par  $\varepsilon^{1-\alpha}$  et en faisant  $\varepsilon \rightarrow 0$ , on obtient  $m = 0$  qui mène à une contradiction puisqu'on a supposé  $m < 0$ . Par conséquent, on a formellement montré que sur la frontière  $h(x) - Du(x) \cdot n(x) = 0$  ou  $u_t - \mathcal{F}(t, x, Du, D^2 u) = 0$ .

### Quelques commentaires :

◊ Il est important de remarquer que ce développement heuristique est valide si  $m$  est assez loin de zéro et nous devons modifier nos arguments très soigneusement à la section lorsque  $m$  est négatif mais petit par rapport à une certaine puissance de  $\varepsilon$ .

◊ A la différence du jeu à l'intérieur du domaine et des stratégies correspondantes, lorsque Helen choisit  $p$  et  $\Gamma$  de façon optimale, elle ne devient pas indifférente au choix que fera Mark. Plus précisément, ce choix dépend de la projection de  $\Delta\hat{x}$  sur l'axe  $n(\bar{x})$ . Nos jeux conserveront systématiquement cette caractéristique. Dans cette perspective, on peut aussi remarquer que lorsque Helen doit choisir les dernières composantes  $\langle \Gamma e_1, e_j \rangle$ ,  $2 \leq j \leq d$  dans la dérivation heuristique, il existe deux choix équivalents donnés par

$$\langle \Gamma e_1, e_j \rangle = \langle D^2 u e_1, e_j \rangle \quad \text{et} \quad \langle \Gamma e_1, e_j \rangle = \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \langle D^2 u e_1, e_j \rangle.$$

Par souci de simplicité, on a retenu le premier choix dans (1.2.52). Cela correspond dans l'analyse au fait qu'il existe deux déplacements équivalents pour Mark lorsque la position de la particule est proche du bord, les deux déplacements étant symétriques suivant la réflexion normale par rapport tels un objet se reflétant dans un miroir.

**2.1.2.2 Enoncé du théorème** Pour présenter notre résultat, nous avons besoin d'introduire deux fonctions relaxées  $\bar{u}$  et  $\underline{u}$ . En suivant la méthode de Barles and Perthame [18], nous posons

$$\bar{u}(x, t) := \limsup_{\substack{y \rightarrow x, y \in \bar{\Omega} \\ t_j \rightarrow t \\ \varepsilon \rightarrow 0}} u^\varepsilon(y, t_j) \quad \text{et} \quad \underline{u}(x, t) := \liminf_{\substack{y \rightarrow x, y \in \bar{\Omega} \\ t_j \rightarrow t \\ \varepsilon \rightarrow 0}} u^\varepsilon(y, t_j), \quad (33)$$

où les temps discrets sont  $t_j = T - j\varepsilon^2$ . En particulier, il est clair par définition que  $\bar{u} \geq \underline{u}$ . Nous allons montrer que sous des hypothèses convenables,  $\bar{u}$  et  $\underline{u}$  sont respectivement sous-solution et sur-solution de (17). Nous pouvons maintenant énoncer notre résultat principal dans le cas parabolique.

**Théorème 7** (Convergence pour le jeu parabolique, D.). *Soit le problème (17) où  $\mathcal{F}$  est elliptique et vérifie (10)–(12),  $g$  et  $h$  sont continues, uniformément bornées, et  $\Omega$  vérifie (H1)–(H2). Supposons que les paramètres  $\alpha, \beta, \gamma$  satisfont les bonnes hypothèses. Alors :*

- $\bar{u}$  et  $\underline{u}$  sont uniformément bornées sur  $\bar{\Omega} \times [t_*, T]$  pour tout  $t_* < T$  et sont respectivement sous-solution de viscosité et sur-solution de viscosité de (17).
- De plus, si l'EDP possède un principe de comparaison (pour les solutions uniformément bornées), alors il en découle que  $u^\varepsilon$  converge localement uniformément vers l'unique solution de viscosité de (17).

**2.1.2.3 Preuve rigoureuse** L'idée sous-jacente de la preuve du théorème 7 repose sur une méthode établie par Barles et Souganidis portant sur les schémas d'approximation de solutions de viscosité d'équations non-linéaires elliptiques et paraboliques [20, Section 2, page 275].

**Proposition 8.** (Barles et Souganidis [20]) *Si un schéma numérique  $S_\varepsilon$  est monotone, stable et consistant, alors la limite semi-relaxée inférieure (resp. supérieure) de la solution  $u^\varepsilon$  du schéma est une sur-solution (resp. sous-solution) de l'EDP. Si de plus, l'EDP possède un principe de comparaison, la solution  $u^\varepsilon$  du schéma converge localement uniformément vers l'unique solution de viscosité de l'EDP lorsque  $\varepsilon \rightarrow 0$ .*

Nous expliquons dès à présent les hypothèses utilisées ci-dessus dans le cas de notre schéma parabolique :

- ◊ *Monotonie.* Elle correspond à la propriété (26). On l'obtient immédiatement comme c'est souvent le cas pour les opérateurs obtenus par des principes de programmation dynamique.
- ◊ *Stabilité.* Pour tout  $\varepsilon > 0$ , la solution  $u^\varepsilon$  est bornée sur  $\bar{\Omega}$  avec une borne indépendante de  $\varepsilon$ .
- ◊ *Consistance.* Elle revient à obtenir des estimations, dites de *consistance*, qui fournissent l'action sur des fonctions régulières du générateur infinitésimal associé au principe de programmation dynamique donné par le jeu. On cherche à contrôler la limite supérieure (resp. inférieure) de l'action de l'opérateur par la limite semi-relaxée supérieure (resp. inférieure) de l'hamiltonien. Nous sommes donc conduits à étudier un développement asymptotique lorsque le paramètre  $\varepsilon \rightarrow 0$  de la quantité

$$S_\varepsilon[x, t, \phi] - \phi(x, t).$$

De plus, on dit que l'EDP possède un *principe de comparaison au sens fort* si :

$$\begin{cases} u \text{ est une sous-solution SCS de l'EDP} \\ v \text{ est une sur-solution SCI de l'EDP} \end{cases} \implies u \leq v \quad \text{sur } \bar{\Omega}.$$



Nous allons montrer que les fonctions  $\bar{u}$  and  $\underline{u}$  données par (33) sont respectivement sous-solution et sur-solution de l'EDP. Si on suppose de plus que l'EDP possède un principe de comparaison, on pourra alors conclure que l'EDP possède une unique solution de viscosité  $\bar{u} = \underline{u}$ . Notre preuve de ce résultat de convergence combine les trois propriétés évoquées ci-dessus. La monotonie étant immédiate, nous verrons rapidement la preuve de la stabilité puis nous allons détailler la preuve de la consistance, ou plutôt des estimations de consistance, qui constitue le point le plus délicat de l'analyse rigoureuse en raison du terme de pénalisation pour la condition de Neumann intervenant près de la frontière.

### Stabilité :

Cette propriété est donnée par le fait que si les conditions au bord sont uniformément bornées, alors  $u^\varepsilon$  reste borné lorsque  $\varepsilon \rightarrow 0$ . Autrement dit, on veut montrer qu'il y a absence d'explosion pour la norme  $L^\infty$ . Pour notre problème, on utilise un argument par comparaison en encadrant la solution  $u^\varepsilon$  du jeu par des barrières inférieures et supérieures prenant en compte la condition de Neumann au voisinage du bord. Pour construire cette fonction barrière, il suffit de considérer une fonction  $\psi$  régulière satisfaisant  $\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1$  (elle est construite par résolution d'un problème elliptique adapté). La figure 3 illustre l'influence d'une condition de Neumann  $h = \pm 1$  dans le cas particulier de l'équation de la chaleur donnée sur un disque.

**Proposition 9.** *Supposons que  $\mathcal{F}$  satisfait (E) et (11)–(12) et que les paramètres du jeu  $\alpha, \beta, \gamma$  sont correctement choisis. Supposons de plus que la donnée finale  $g$  est bornée. Alors il existe une constante  $s = s(\|\psi\|_{C_b^2(\bar{\Omega})})$ , indépendante de  $\varepsilon$ , telle que, pour tout  $x \in \bar{\Omega}$  et  $t < T$ , on a :*

$$-(\|g\|_{L^\infty(\bar{\Omega})} + \|\psi\|_{L^\infty(\bar{\Omega})})s^{T-t} - \psi(x) \leq v^\varepsilon(x, t) \leq u^\varepsilon(x, t) \leq (\|g\|_{L^\infty(\bar{\Omega})} + \|\psi\|_{L^\infty(\bar{\Omega})})s^{T-t} + \psi(x).$$

### Consistance :

Pour notre opérateur parabolique, on examine

$$\begin{aligned} S_\varepsilon[x, t, \phi] - \phi(x) &= \max_{p, \Gamma} \min_{\Delta \hat{x}} \left[ (D\phi(x) - p) \cdot \Delta \hat{x} + \frac{1}{2} \langle D^2 \phi(x) \Delta x, \Delta x \rangle \right. \\ &\quad \left. + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\} - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 f(t, x, p, \Gamma) \right] + o(\varepsilon^2). \end{aligned} \quad (34)$$

Dans le cas de notre dynamique, le jeu continue toujours, il ne peut pas s'arrêter contrairement au jeu correspondant à la condition de Dirichlet. Dans le cas de ce dernier jeu, il n'existe pas d'estimée de consistance dépendant de la condition de Dirichlet tandis qu'ici l'estimée de consistance peut dépendre de la condition de Neumann. Il faudra distinguer plusieurs cas suivant la valeur du petit paramètre  $\varepsilon > 0$ , l'amplitude et le signe de la pénalisation due au terme de Neumann et la distance de la particule au bord car il y a compétition entre le terme de pénalisation et le terme correspondant à l'estimation usuelle de la consistance. Le cas limite correspond au cas où  $\text{dist}(x, \partial\Omega) = \varepsilon^{1-\alpha}$  pour lequel la pénalisation n'intervient pas. Plus généralement, si  $x$  satisfait  $\text{dist}(x, \partial\Omega) \geq \varepsilon^{1-\alpha}$ , l'estimation usuelle obtenue dans [88] est conservée :

$$\lim_{\varepsilon \rightarrow 0} \frac{S_\varepsilon[x, t, \phi] - \phi(x, t)}{\varepsilon^2} = -\mathcal{F}(t, x, D\phi(x, t), D^2\phi(x, t)).$$

Il reste à traiter le cas  $\text{dist}(x, \partial\Omega) \geq \varepsilon^{1-\alpha}$ . Si on se rappelle la définition de sous-solution et sur-solution de viscosité pour le problème de Neumann, il est naturel que les relaxations semi-inférieure et supérieure vont faire intervenir d'une part la condition de Neumann et d'autre part la non-linéarité habituelle et l'estimation classique donnée ci-dessus peut dégénérer.

Nous donnons l'objet-clé pour réaliser l'analyse mathématique rigoureuse de la consistance au voisinage du bord pour la condition de Neumann. Nous allons introduire des applications donnant des bornes inférieure et supérieure sur le terme correspondant à la condition de Neumann pour une fonction régulière et un point proche de la frontière. Plus précisément, on définit des applications  $m_\varepsilon$  et  $M_\varepsilon$ , pour  $\phi \in C^1(\bar{\Omega})$

et  $x$  tel que  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$ , données par

$$m_\varepsilon^x[\phi] := \inf_{\substack{x+\Delta\hat{x} \notin \Omega \\ \Delta\hat{x}}} \{h(x+\Delta x) - D\phi(x) \cdot n(x+\Delta x)\}, \quad (35)$$

$$M_\varepsilon^x[\phi] := \sup_{\substack{x+\Delta\hat{x} \notin \Omega \\ \Delta\hat{x}}} \{h(x+\Delta x) - D\phi(x) \cdot n(x+\Delta x)\}, \quad (36)$$

où  $\Delta\hat{x}$  est restreint par (21) et détermine  $\Delta x$  par la relation (23). Puisque  $h$  est continue, la propriété

$$\lim_{\varepsilon \rightarrow 0} m_\varepsilon^x[\phi] = \lim_{\varepsilon \rightarrow 0} M_\varepsilon^x[\phi] = h(x) - D\phi(x) \cdot n(x), \quad \text{pour } x \in \partial\Omega,$$

est immédiate et explique simplement pourquoi cette approche bien adaptée au cadre des solutions de viscosité pour étudier la *convergence* de la solution du jeu vers la solution de l'EDP. A partir de ces deux indicateurs, on définit alors des stratégies pour Helen  $p_{\text{opt}}^m(x)$ ,  $p_{\text{opt}}^M(x) \in \mathbb{R}^d$  et  $\Gamma_{\text{opt}}(x) \in \mathcal{S}^d$  dans une base orthonormale  $\mathcal{B} = (e_1 = n(\bar{x}), e_2, \dots, e_d)$  respectivement données par

$$p_{\text{opt}}^m(x) := D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m_\varepsilon^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] n(\bar{x}), \quad (37)$$

$$p_{\text{opt}}^M(x) := D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) M_\varepsilon^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] n(\bar{x}), \quad (38)$$

et

$$\Gamma_{\text{opt}}(x) := D^2\phi(x) + \left[ \frac{1}{2} \left( -1 + \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] E_{11}, \quad (39)$$

où  $m_\varepsilon^x[\phi]$  et  $M_\varepsilon^x[\phi]$  sont données par (35) et (36) et  $E_{11}$  représente la matrice qui vaut 1 pour la case  $(1, 1)$  et 0 partout ailleurs dans la base  $\mathcal{B}$ .

Pour obtenir des bornes supérieures précises sur (34), nous introduisons successivement deux paramètres supplémentaires  $\rho, \kappa > 0$  tels que

$$1 - \alpha < \rho < \min \left( 1 - \frac{\gamma(r-1)}{2}, 2 - 2\alpha - \gamma \right), \quad (40)$$

et

$$\gamma + \rho - (1 - \alpha) < \kappa < 1 - \alpha. \quad (41)$$

Ces coefficients sont bien définis en vertu des relations (18) et (19).

**Proposition 10** (Développement asymptotique de l'opérateur  $S_\varepsilon$ ). *Supposons que  $\mathcal{F}$  satisfait (E) et (11)–(12) et que les paramètres du jeu  $\alpha, \beta, \gamma$  sont correctement choisis. Soient  $p_{\text{opt}}^m, p_{\text{opt}}^M$  et  $\Gamma_{\text{opt}}$  respectivement définis dans une base orthonormale  $(e_1 = n(\bar{x}), e_2, \dots, e_d)$  par (37), (38) et (39). Pour tout  $x, t$  et toute fonction régulière  $\phi$  définie au voisinage de  $x$ ,  $S_\varepsilon[x, t, \phi]$  étant défini par (25), nous distinguons deux cas pour la minoration :*

i. *Grand bonus : si  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$  ou  $m_\varepsilon^x[\phi] > \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , alors*

$$S_\varepsilon[x, t, \phi] - \phi(x) \geq -\varepsilon^2 \mathcal{F}(t, x, D\phi(x), D^2\phi(x)).$$

ii. *Pénalité ou petit bonus : si  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  et  $m_\varepsilon^x[\phi] \leq \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , alors*

$$S_\varepsilon[x, t, \phi] - \phi(x) \geq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) (sm_\varepsilon^x[\phi] - 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}) - \varepsilon^2 \mathcal{F}(t, x, p_{\text{opt}}^m(x), \Gamma_{\text{opt}}(x)),$$

avec  $s = -1$  si  $m_\varepsilon^x[\phi] \geq 0$  et  $s = 3$  si  $m_\varepsilon^x[\phi] < 0$ .

Soient  $\rho, \kappa > 0$  satisfaisant (40)–(41). Nous distinguons quatre cas pour la majoration :

i. *Grand bonus : si  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  et  $M_\varepsilon^x[\phi] > \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , alors*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq 3(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2 \mathcal{F}(t, x, p_{\text{opt}}^M(x), \Gamma_{\text{opt}}(x)) + o(\varepsilon^2).$$

- ii. Loin de la frontière avec un petit bonus : si  $\varepsilon^{1-\alpha} - \varepsilon^\rho \leq \text{dist}(x) \leq \varepsilon^{1-\alpha}$  et  $M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , ou si  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ , alors

$$S_\varepsilon[x, t, \phi] - \phi(x) \leq -\varepsilon^2 \mathcal{F}(t, x, D\phi(x), D^2\phi(x)) + o(\varepsilon^2).$$

- iii. Près de la frontière avec un petit bonus/pénalité : si  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  et  $-\varepsilon^{1-\alpha-\kappa} \leq M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , alors

$$S_\varepsilon[x, t, \phi] - \phi(x) \leq -\varepsilon^2 \mathcal{F}(t, x, D\phi(x), D^2\phi(x) + C_1 I_d) + o(\varepsilon^2),$$

$$\text{avec } C_1 = \frac{20}{3}\|D^2\phi(x)\| \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right).$$

- iv. Près de la frontière avec une grande pénalité : si  $d(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  et  $M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}$ , alors

$$S_\varepsilon[x, t, \phi] - \phi(x) \leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2 \min_{p \in B(p_{opt}^M(x), r_{opt}(x))} \mathcal{F}(t, x, p, \Gamma_{opt}(x)) + o(\varepsilon^2), \quad (42)$$

$$\text{avec } r_{opt}(x) := 3 \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\phi]|.$$

De plus, les constantes implicites dans le terme d'erreur sont uniformes lorsque  $(x, t)$  appartient à un compact de  $\overline{\Omega} \times \mathbb{R}$ .

Cette proposition fournit les développements asymptotiques d'une borne inférieure et d'une borne supérieure pour l'opérateur de programmation dynamique en effectuant une séparation de cas suivant la valeur des indicateurs  $m_\varepsilon^x[\phi]$  et  $M_\varepsilon^x[\phi]$  et la distance  $\text{dist}(x)$  de la particule au bord. Par la forme en min sur le déplacement, l'étude la plus délicate est celle de la borne supérieure car nous devons choisir de façon pertinente la valeur du déplacement suivant la valeur de  $p$  et  $\Gamma$ . Le but est d'obtenir des estimations précises sur (34) dans les trois cas suivants :  $M_\varepsilon^x[\phi]$  très négatif,  $M_\varepsilon^x[\phi]$  très positif et  $M_\varepsilon^x[\phi]$  proche de zéro, les bornes séparant les différents cas dépendants de certaines puissances de  $\varepsilon$ . Nous avons expliqué formellement à la section 0.2.1.2.1 que le premier cas est favorable à Mark puisque Helen peut subir une pénalité importante si Mark choisit de traverser la frontière. Au contraire, le second cas est favorable à Helen parce qu'elle peut recevoir un grand coupon si la frontière est traversée. Dans le dernier cas, la frontière est transparente (penser à  $M_\varepsilon^x[\phi] = 0$ ) et la pénalisation due à la frontière doit être analysée à travers les termes du second ordre.

On peut alors montrer la propriété de convergence.

**Proposition 11.** *Supposons que  $\mathcal{F}$ ,  $\Omega$  et les paramètres  $\alpha$ ,  $\beta$ ,  $\gamma$  satisfont les hypothèses du théorème 7. Supposons en plus que  $\bar{u}$  et  $\underline{u}$  sont finies au voisinage de  $x_0$  et pour tout  $t \leq T$  près de  $t_0$ . Alors,*

- Cas 1.  $t_0 < T$  et  $x_0 \in \Omega$  :  $\bar{u}$  est une sous-solution de viscosité de (17) en  $(x_0, t_0)$  et  $\underline{u}$  est une sur-solution de (17) en  $(x_0, t_0)$  (i.e., chacune satisfait la sous-partie (P1) pertinente de la définition 2 en  $(x_0, t_0)$ ).
- Cas 2.  $t_0 < T$  et  $x_0 \in \partial\Omega$  :  $\bar{u}$  est une sous-solution de viscosité de (17) en  $(x_0, t_0)$  et  $\underline{u}$  est une sur-solution de (17) en  $(x_0, t_0)$  (i.e., chacune satisfait la sous-partie (P2) pertinente de la définition 2 en  $(x_0, t_0)$ ).
- Cas 3.  $t_0 = T$  :  $\bar{u}(x_0, T) = g(x_0)$  et  $\underline{u}(x_0, T) = g(x_0)$  (en particulier, chacune satisfait la sous-partie (P3) pertinente de la définition 2 en  $(x_0, t_0)$ ).

En particulier, si  $\bar{u}$  et  $\underline{u}$  sont finies pour tout  $x \in \overline{\Omega}$  et  $t_* < t \leq T$ , alors elles sont respectivement sous-solution et sur-solution de viscosité de (17) sur cet intervalle de temps.

Pour établir ce résultat de convergence, on doit raisonner sur toutes les sous-suites possibles de points  $(y, s, \varepsilon)$  convergeant vers  $(x, t, 0)$ . En prenant soin de distinguer les différents cas possibles pour les estimations de consistance données par la proposition 10, combinées avec la propriété de stabilité établie par la proposition et 9 et la propriété de monotonie, on montre que la limite semi-relaxée supérieure (resp. inférieure) vérifie la définition de sous-solution (resp. sur-solution) de viscosité.

On peut alors déduire le théorème 7, le second point du théorème découlant du principe de comparaison associé à l'EDP.

### 2.1.3 Le cas elliptique

Ce paragraphe explique comment notre jeu peut être utilisé pour résoudre des problèmes stationnaires présentant des conditions au bord de Neumann. Le cadre est similaire à la situation parabolique mais un nouvel aspect apparaît : nous devons introduire un facteur d'actualisation comme dans [88], afin de garantir que la fonction valeur d'Helen est finie. Par conséquent, on s'intéresse à

$$\begin{cases} \mathcal{F}(x, u, Du, D^2u) + \lambda u = 0, & \text{sur } \Omega, \\ \langle Du, n \rangle = h, & \text{sur } \partial\Omega, \end{cases} \quad (43)$$

où le domaine  $\Omega$  est de classe  $C^2$  et satisfait simultanément les conditions uniformes de boule extérieure et intérieure données à la Définition 1. La constante  $\lambda$ , qui joue le rôle d'un taux d'intérêt, doit être positif, et assez grand de sorte que (13) est vérifiée. Observons aussi que si  $\mathcal{F}$  est indépendant de  $z$ , alors on peut choisir librement le paramètre  $\lambda$ . La condition au bord  $h$  est supposée être une fonction continue et bornée sur le bord  $\partial\Omega$ .

Pour le problème stationnaire (43), les définitions de sous-solution et sur-solution de l'équation avec condition de Neumann sont similaires au cas dépendant du temps données à la définition 2. Elles sont données précisément au chapitre 1.

Nous présentons maintenant le jeu. La principale différence avec le cas elliptique est la présence du facteur d'actualisation  $\lambda$ . En plus des paramètres  $\alpha, \beta, \gamma$  déjà introduits, nous avons besoin d'introduire dans le cas stationnaire deux nouveaux paramètres positifs,  $m$  et  $M$ , et une fonction  $\psi$  qui est  $C_b^2(\overline{\Omega})$  telle que

$$\frac{\partial\psi}{\partial n} = \|h\|_\infty + 1 \quad \text{sur } \partial\Omega. \quad (44)$$

Pour construire une telle fonction, on se ramène à la construction d'une fonction  $\psi_1$  qui est  $C_b^2(\overline{\Omega})$  et satisfait  $\frac{\partial\psi_1}{\partial n} = 1$  sur le bord et on pose  $\psi := (\|h\|_\infty + 1)\psi_1$ . A partir de  $m$  et  $\psi$ , on construit une fonction positive  $\chi$  définie par

$$\chi(x) = m + \|\psi\|_{L^\infty(\overline{\Omega})} + \psi(x). \quad (45)$$

Le paramètre  $M$  sert à borner le score tandis que la fonction  $\chi$  détermine ce qui arrive lorsque la borne est atteinte ou dépassée. A la fin de l'analyse, on devra finalement choisir  $m$  tel que  $m + 2\|\psi\|_{L^\infty} = M - 1$  et demander également que  $M$  est suffisamment grand. A l'instar des choix pour  $\alpha, \beta, \gamma$ , les paramètres  $M, m$  et la fonction  $\psi$  sont utilisés pour définir le jeu mais ils n'influencent pas l'EDP associée.

La définition de  $U^\varepsilon(x, z)$  pour  $x \in \overline{\Omega}$  utilise un jeu similaire à celui de la section précédente :

**Etape 1 :** Au début, à l'instant  $t_0 = 0$ , le prix de l'action est  $x_0 = x$  et la dette d'Helen est  $z_0 = z$ .

**Etape 2 :** Supposons, à l'instant  $t_j = j\varepsilon^2$ , le prix de l'action est  $x_j$  et la dette d'Helen est  $z_j$  avec  $|z_j| < M$ . Alors Helen choisit un vecteur  $p_j \in \mathbb{R}^d$  et une matrice  $\Gamma_j \in \mathcal{S}_d$ , dont les tailles sont restreintes par (20). Connaissant ces choix, Mark détermine le prochain prix de l'action  $x_{j+1} = x_j + \Delta x$  de sorte à dégrader/diminuer le résultat d'Helen. L'incrément  $\Delta x$  permet de modéliser la réflexion exactement de la même sorte que dans le cas précédent. Mark choisit un point intermédiaire  $\hat{x}_{j+1} = x_j + \Delta\hat{x}_j \in \mathbb{R}^d$  tel que

$$\|\Delta\hat{x}_j\| \leq \varepsilon^{1-\alpha}.$$

Cette position  $\hat{x}_{j+1}$  détermine la nouvelle position  $x_{j+1} = x_j + \Delta x_j$  à l'instant  $t_{j+1}$  par

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}).$$

Helen subit à l'instant  $t_j$  une perte

$$\delta_j = p_j \cdot \Delta\hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta\hat{x}_j, \Delta\hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(x_j, z_j, p_j, \Gamma_j) - \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j). \quad (46)$$

En conséquence, sa dette à l'instant  $t_{j+1} = t_j + \varepsilon^2$  devient

$$z_{j+1} = e^{\lambda\varepsilon^2} (z_j + \delta_j),$$

où le facteur  $e^{\lambda\varepsilon^2}$  prend en compte ses intérêts de paiement.

**Etape 3 :** Si  $z_{j+1} \geq M$ , alors le jeu se termine, et Helen paie une “pénalité de fin par grande dette” de valeur  $e^{\lambda\varepsilon^2}(\chi(x_j) - \delta_j)$  à l’instant  $t_{j+1}$ . De manière analogue, si  $z_{j+1} \leq -M$ , alors le jeu se termine, et Helen reçoit un “bonus de fin par grande richesse” de valeur  $e^{\lambda\varepsilon^2}(\chi(x_j) + \delta_j)$  à l’instant  $t_{j+1}$ . Si le jeu se termine de cette façon, on appelle  $t_{j+1}$  the “ending index”  $t_K$ .

**Etape 4 :** Si le jeu n’est pas terminé, Helen et Mark répètent cette procédure à l’instant  $t_{j+1} = t_j + \varepsilon^2$ . Si le jeu ne s’arrête jamais, le “ending index”  $t_K$  est  $+\infty$ .

On observe que le jeu elliptique est stationnaire car rien ne distingue l’instant 0 et on obtient un principe de programmation dynamique sur une certaine fonction  $U^\varepsilon(x, z)$  qui est que pour  $|z| < M$ ,

$$U^\varepsilon(x, z) = \sup_{p, \Gamma} \inf_{\Delta\hat{x}} \begin{cases} e^{-\lambda\varepsilon^2} U^\varepsilon(x', z'), & \text{si } |z'| < M, \\ -z - \chi(x), & \text{si } z' \geq M, \\ -z + \chi(x), & \text{si } z' \leq -M, \end{cases} \quad (47)$$

où  $x' = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x})$  et  $z' = e^{\lambda\varepsilon^2}(z + \delta)$ , avec  $\delta$  défini comme dans (46). Ici  $p$ ,  $\Gamma$  et  $\Delta\hat{x}$  sont restreints comme d’habitude par (20)–(21), et on écrit sup / inf plutôt que max / min puisqu’il n’est plus clair que les valeurs optimales sont réalisées (puisque le membre de droite est maintenant une fonction discontinue de  $p$ ,  $\Gamma$  et  $\Delta\hat{x}$ ). La définition des fonctions  $u^\varepsilon$  et  $v^\varepsilon$  d’intérêt est légèrement différente que celle prise dans le cas parabolique :

$$u^\varepsilon(x_0) := \sup\{z_0 \in \mathbb{R} : U^\varepsilon(x_0, z_0) > 0\} \quad \text{et} \quad v^\varepsilon(x_0) := \inf\{z_0 \in \mathbb{R} : U^\varepsilon(x_0, z_0) < 0\}.$$

Cette définition suit celle suivie dans le cadre parabolique général incluant la dépendance de la non-linéarité en  $u$  (rappelée à la section 0.1.3.2) hormis le fait que les inégalités sont strictes au lieu d’être larges. Le changement principal dans le cadre elliptique par rapport au paragraphe 0.2.1.2 est que nous obtenons deux inéquations de programmation dynamique sur les fonctions  $u^\varepsilon$  et  $v^\varepsilon$ . Celles-ci sont données, pour  $x \in \overline{\Omega}$ ,  $\varepsilon > 0$  et  $u^\varepsilon(x), v^\varepsilon(x) \in (-M, M)$  convenablement choisis, par

$$u^\varepsilon(x) \leq S_\varepsilon[x, u^\varepsilon(x), u^\varepsilon] \quad (48)$$

$$v^\varepsilon(x) \geq S_\varepsilon[x, u^\varepsilon(x), v^\varepsilon], \quad (49)$$

où  $S_\varepsilon[x, z, \phi]$  est défini pour  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$  et toute fonction continue  $\phi : \overline{\Omega} \rightarrow \mathbb{R}$  par est défini par

$$S_\varepsilon[x, z, \phi] = \sup_{p, \Gamma} \inf_{\Delta\hat{x}} \left[ e^{-\lambda\varepsilon^2} \phi(x + \Delta x) - \left( p \cdot \Delta\hat{x} + \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, z, p, \Gamma) - \|\Delta\hat{x} - \Delta x\| h(x + \Delta x) \right) \right].$$

La nature précise des dépendances entre  $u^\varepsilon(x)$ , avec  $M$  et le paramètre  $\varepsilon > 0$  est donnée au chapitre 1, section 1.2.3. L’EDP (43) est l’équation d’Hamilton-Jacobi-Bellman associée aux inégalités de programmation dynamique (48)–(49), par le développement de Taylor habituel si  $-M < v^\varepsilon \approx u^\varepsilon < M$ . Pour énoncer notre théorème dans le cas elliptique, nous avons besoin d’introduire les semi-limites relaxées suivantes, définies pour tout  $x \in \overline{\Omega}$ ,

$$\overline{u}(x) := \limsup_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} u^\varepsilon(y) \quad \text{and} \quad \underline{v}(x) := \liminf_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} v^\varepsilon(y), \quad (50)$$

en considérant les limites telles que  $y$  tend vers  $x$  sur  $\overline{\Omega}$ , ce qui est effectivement possible car  $u^\varepsilon$  et  $v^\varepsilon$  sont définies sur  $\overline{\Omega}$ .

**Théorème 12** (Convergence pour le jeu elliptique, D.). *Considérons le problème stationnaire (43) où  $\mathcal{F}$  satisfait (E) et (13)–(15)  $g$  et  $h$  sont continues, uniformément bornées et  $\Omega$  vérifie (H1)–(H2). Supposons que les paramètres  $\alpha$ ,  $\beta$ ,  $\gamma$  du jeu remplissent les conditions adéquates,  $\psi \in C^2(\overline{\Omega})$  satisfait (44),  $\chi \in C^2(\overline{\Omega})$  est défini par (45),  $M$  est suffisamment grand et  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ . Alors :*

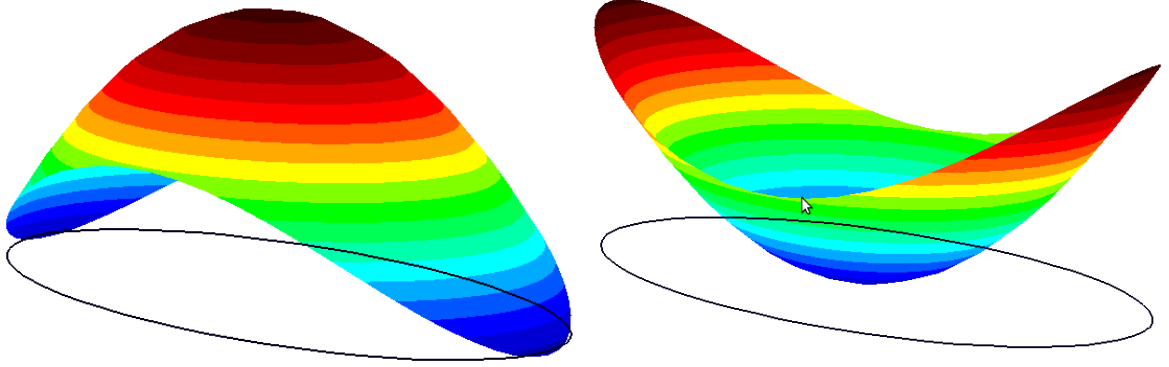


Figure 3 – Equation de la chaleur sur un disque avec condition de Neumann :  $h = -1$  à gauche,  $h = 1$  à droite.

- Les fonctions  $u^\varepsilon$  et  $v^\varepsilon$  sont bien définies pour  $\varepsilon$  suffisamment petit, et satisfont  $|u^\varepsilon| \leq \chi$  et  $|v^\varepsilon| \leq \chi$ . Leurs limites semi-relaxées  $\bar{u}$  et  $\underline{v}$  sont respectivement sous-solution de viscosité et sur-solution de viscosité de (43).
- Si en plus, on a  $\underline{v} \leq \bar{u}$  et que l'EDP possède un principe de comparaison, alors  $u^\varepsilon$  et  $v^\varepsilon$  convergent localement uniformément sur  $\bar{\Omega}$  vers l'unique solution de viscosité de (43).

L'un des changements importants dans le cas elliptique concerne l'existence de la fonction  $u^\varepsilon$ . En effet, nous devons maintenant appliquer un théorème de point fixe de Picard pour établir l'existence de la fonction  $U^\varepsilon$  dont découlera ensuite l'existence de la fonction  $u^\varepsilon$ . Par ailleurs, dans le cas d'une non-linéarité  $\mathcal{F}$  dépendant de  $u$ , il n'est plus clair par définition des sous-limites relaxées que  $u^\varepsilon \leq v^\varepsilon$ . Le reste de la preuve du résultat de convergence est similaire à celle du cas parabolique. C'est une conséquence immédiate des estimations de consistance, qui ont une forme analogue au cas parabolique, et d'une estimation de stabilité. Ces dernières sont données rigoureusement au chapitre 1 dans les propositions 1.3.6 et 1.5.5. Une condition suffisante pour garantir que  $\underline{v} \leq \bar{u}$  est que  $\mathcal{F}$  est croissante par rapport à  $z$ . Comme mentionné dans [88], des conditions suffisantes sur l'EDP pour présenter un principe de comparaison peuvent être trouvées par exemple à la section 5 du "User's guide" [45] et aussi dans [13] et [19].

#### 2.1.4 Quelques généralisations et questions ouvertes

Les résultats précédents peuvent être généralisés au problème oblique ou au problème mixte Dirichlet-Neumann. Nous énonçons ici rapidement ces résultats.

##### 2.1.4.1 Le cas mixte Dirichlet-Neumann

Considérons l'équation elliptique donnée par

$$\begin{cases} \mathcal{F}(x, u, Du, D^2u) + \lambda u = 0, & \text{dans } \Omega, \\ u = g, & \text{sur } \Upsilon_D, \\ \frac{\partial u}{\partial n} = h, & \text{sur } \Upsilon_N, \end{cases} \quad (51)$$

où on considère une partition  $\Upsilon_D \cup \Upsilon_N = \partial\Omega$  du bord  $\partial\Omega$  avec  $\Upsilon_D$  fermé non-vidé et  $\Upsilon_N$  de classe  $C^2$ . Par ailleurs, on impose d'une part une condition de boule extérieure uniforme sur  $\Omega$  et d'autre part une condition de boule intérieure uniforme sur un voisinage de  $\Upsilon_N$  (voir définition 1). De manière similaire au cas elliptique, on introduira d'une fonction  $\psi$  de classe  $C_b^2(\bar{\Omega})$  telle que

$$\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1 \quad \text{on } \Upsilon_N. \quad (52)$$

et on lui associe une fonction  $\chi$  de la même façon que dans le cas elliptique avec condition de Neumann. Cette généralisation découle directement de l'interprétation classique des dynamiques vis-à-vis des conditions aux limites. En effet, la dynamique correspondant à une condition de Dirichlet consiste à s'arrêter lorsque la particule franchit la frontière (voir [88] pour une analyse détaillée de cette condition pour le jeu de Mark et Helen). Par conséquent, on fait continuer le jeu si la projection de la position sur le bord du domaine appartient à la partie  $\Upsilon_N$  correspondant à la condition de Neumann et au contraire, on fait arrêter le jeu si cette projection appartient à la partie  $\Upsilon_D$  associée à la condition de Dirichlet.

**Théorème 13** (Convergence pour l'EDP elliptique mixte Dirichlet-Neumann, D.). *Soit le problème stationnaire (51) avec condition au bord avec  $\mathcal{F}$  elliptique et satisfaisant (13)–(15),  $g$  et  $h$  sont continues, uniformément bornées et  $\Omega$  vérifie (H1) et satisfait la condition uniforme de boule extérieure sur  $\overline{\Omega}$  et la condition uniforme de boule intérieure sur un voisinage de  $\Upsilon_N$ . Supposons que les paramètres  $\alpha, \beta, \gamma$  du jeu sont convenablement choisis,  $\psi \in C_b^2(\overline{\Omega})$  satisfait (52),  $\chi \in C^2(\overline{\Omega})$  est définie correctement,  $M$  est assez grand, et  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ . Alors :*

- $u^\varepsilon$  et  $v^\varepsilon$  sont bien définis pour  $\varepsilon$  assez petit et satisfont  $|u^\varepsilon| \leq \chi$  et  $|v^\varepsilon| \leq \chi$ . Leurs limites semi-relaxées  $\bar{u}$  et  $\underline{v}$  sont respectivement sous-solution et sur-solution de viscosité de (51).
- Si de plus, on a  $\underline{v} \leq \bar{u}$  et l'EDP possède un principe de comparaison, alors  $u^\varepsilon$  et  $v^\varepsilon$  convergent localement uniformément sur  $\overline{\Omega}$  vers l'unique solution de viscosité de (51).

**2.1.4.2 Le cas oblique** Nous voulons ici construire un jeu qui représente l'EDP parabolique non-linéaire associée à une condition oblique  $h$  et une donnée finale  $g$  donnée par

$$\begin{cases} \partial_t u - \mathcal{F}(t, x, u, Du, D^2u) = 0, & \text{pour } x \in \Omega \text{ et } t < T, \\ \frac{\partial u}{\partial \varsigma}(x, t) = h(x), & \text{pour } x \in \partial\Omega \text{ et } t < T, \\ u(x, T) = g(x), & \text{pour } x \in \overline{\Omega}, \end{cases} \quad (53)$$

où  $\varsigma$  définit un champ de vecteur sur  $\partial\Omega$  régulier (disons  $C^2$ ) et dirigé vers l'extérieur tel que

$$\langle \varsigma(x), n(x) \rangle \geq \theta > 0 \quad \text{pour tout } x \in \partial\Omega. \quad (54)$$

Comme d'habitude, le domaine  $\Omega$  est de classe  $C^2$  au bord et satisfait simultanément les conditions uniformes de boule extérieure et intérieure. En considérant la distance induite par le champ de vecteurs  $\varsigma$  au voisinage du bord, il est possible de définir une projection  $\text{proj}_\Omega^\varsigma$  suivant ce champ de vecteurs sur l'intérieur du domaine dans un voisinage du bord, ce qui fournit un jeu correspondant à la condition oblique. Pour l'analyse, on définit alors, pour des fonctions régulières, des indicateurs  $m_{\varsigma, \varepsilon}^x$ ,  $M_{\varsigma, \varepsilon}^x$  et des stratégies  $p_{\text{opt}}^{m_\varsigma}$ ,  $p_{\text{opt}}^{M_\varsigma}$  et  $\Gamma_{\text{opt}}^\varsigma$  associés à cette métrique oblique, qui sont analogues aux indicateurs donnés par (35)–(39) dans le cas d'une réflexion normale sur le bord. Pour la stabilité, nous avons aussi besoin d'introduire une fonction bornée et de classe  $C^2$ ,

$$\frac{\partial \psi}{\partial \varsigma}(x) = \|h\|_{L^\infty} + 1 \quad \text{on } \partial\Omega.$$

On obtient alors un théorème de convergence qui possède exactement le même énoncé que le théorème 7 sous l'hypothèse supplémentaire (54) sur le champ de vecteurs. Il est énoncé à la section 1.6.2 du chapitre 1.

**2.1.4.3 Questions ouvertes et perspectives** D'autres conditions aux limites de type Neumann peuvent être envisagées, parmi lesquelles :

- ◊ Les conditions de capillarité :

$$\frac{\partial u}{\partial n} = \theta \sqrt{1 + |Du|^2} \quad \text{sur } \partial\Omega,$$

où  $\theta$  est une fonction continue sur  $\partial\Omega$  vérifiant  $|\theta| < 1$  sur  $\partial\Omega$ .

◊ Les conditions intervenant dans le contrôle de processus réfléchis :

$$\sup_{v \in V} \{\gamma(x, v) \cdot Du + \mu u - g(x, v)\} = 0 \quad \text{sur } \partial\Omega.$$

où  $\gamma(x, v) \cdot n(x) \geq \nu > 0$  sur  $\partial U$ , pour tout  $v \in V$  ( $\nu$  indépendant de  $v$ ).

De plus, nous avons considéré l'hypothèse que le domaine non-borné est  $C^2$  et vérifie des conditions de boule intérieure et extérieure uniformes. On a ainsi obtenu une fonction distance suffisamment régulière. On peut chercher des conditions optimales sur la régularité du domaine afin d'obtenir le résultat de convergence du jeu.

## 2.2 Résultats de régularité parabolique

Cette partie est détaillée au chapitre 2 et correspond à l'article [51]. L'objectif est ici d'obtenir des résultats de régularité associés à l'équation parabolique donnée par

$$\partial_t u + \mathcal{F}(D^2 u) = g \quad \text{sur } \Omega \subseteq \mathbb{R}^d \times (0, \infty), \quad (55)$$

pour un opérateur  $\mathcal{F}$  uniformément elliptique et  $g \in C^{0,1}(\Omega)$ . Nous allons d'abord énoncer les résultats obtenus dans la section 0.2.2.1 nous verrons quelques idées des preuves.

### 2.2.1 Théorèmes d'estimation $W^{3,\varepsilon}$ parabolique et de régularité partielle parabolique

Récemment, Armstrong, Smart et Silvestre [2] ont obtenu un résultat de régularité partielle pour les solutions de viscosité de l'équation

$$\mathcal{F}(D^2 u) = 0 \quad \text{sur } U \subseteq \mathbb{R}^d,$$

où l'opérateur  $\mathcal{F}$  est uniformément elliptique, de classe  $C^1$  et possède une différentielle qui est supposée uniformément continue. Ce résultat nécessite d'obtenir au préalable et sous la seule hypothèse d'uniforme ellipticité une estimation en mesure sur l'approximation en termes de développement de Taylor à l'ordre 3 de la solution de viscosité de l'équation. Cette estimation sera appelée *estimation  $W^{3,\varepsilon}$  parabolique*.

Dans un premier temps, nous allons nous intéresser à l'estimation  $W^{3,\varepsilon}$  parabolique qui est un résultat intéressant par lui-même. Dans un second temps et comme première application, nous obtiendrons ensuite le théorème de régularité partielle pour les équations paraboliques avec uniforme ellipticité.

Pour énoncer notre estimation, nous avons besoin d'introduire une définition. Pour  $u : \Omega \rightarrow \mathbb{R}$ , on définit l'application parabolique

$$\begin{aligned} \Psi(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists (b, p, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}_d \text{ tel que } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. |u(y, s) - u(x, t) - b(s - t) - p \cdot (y - x) - \frac{1}{2}(y - x) \cdot M(y - x)| \leq \frac{1}{6}A \left( |x - y|^3 + |s - t|^{3/2} \right) \right\}. \end{aligned} \quad (56)$$

La quantité  $\Psi(u, \Omega)(x, t)$  est définie en termes de polynômes cubiques – d'ordre 3 en espace et d'ordre 3/2 en temps suivant le scaling parabolique – qui touchent  $u$  au point  $(x, t)$  et qui sont respectivement au-dessus ou en-dessous de  $u$  sur le domaine  $\Omega$  tout entier en espace mais restreint aux temps inférieurs ou égaux à  $t$ . Cette quantité respecte une formulation rétrograde en temps, qui est naturelle si on se rappelle la définition du bord parabolique d'un cylindre parabolique donnée par

$$\partial_p(U \times (s, t)) := U \times \{s\} \cup \partial U \times (s, t),$$

ainsi que la forme du principe de comparaison parabolique (voir figure 4). Nous donnons ensuite les conditions de contact par-dessus et dessous dans le cas parabolique.



Soit  $\Omega \subseteq \mathbb{R}^{d+1}$ ,  $u \in \text{LSC}(\Omega)$  et  $v \in \text{USC}(\Omega)$ . On dit que  $v$  touche  $u$  par-dessous en  $(x, t) \in \Omega$  si

$$\begin{cases} v(z, \tau) \leq u(z, \tau), & \text{pour } (z, \tau) \in \Omega \text{ et } \tau \leq t, \\ v(x, t) = u(x, t). \end{cases}$$

On dit que  $u$  touche  $v$  par-dessus en  $(x, t) \in \Omega$  si  $v$  touche  $u$  en  $(x, t) \in \Omega$  par-dessous.

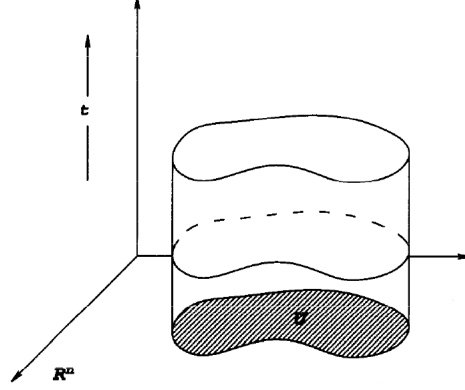


Figure 4 – Principe de comparaison dans le cas parabolique.

De plus, si la fonction  $u$  est suffisamment régulière :  $C^2$  en espace et  $C^1$  en temps, le triplet  $(b, p, M)$  est unique et est donné par :

$$(b, p, M) = (\partial_t u, Du, D^2 u).$$

Cette situation n'est cependant pas systématique car une solution de viscosité de (55) pour un opérateur  $\mathcal{F}$  uniformément elliptique est seulement de classe  $C^{1,\alpha}$  (voir proposition 6). L'énoncé de notre estimation principale est donné par le théorème suivant.

**Théorème 14** (Estimation  $W^{3,\varepsilon}$  parabolique, D.). *Supposons que l'opérateur  $\mathcal{F}$  satisfait (F1),  $g \in C^{0,1}(Q_1)$  et  $u \in C(Q_1)$  est une solution de (55) sur  $Q_1$ . Alors il existe deux constantes universelles  $C, \varepsilon > 0$  telles que, pour tout  $\kappa > 0$ ,*

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(u, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u| + |\mathcal{F}(0)| + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon}.$$

Le résultat analogue elliptique du théorème 14 a été utilisé par exemple pour obtenir des estimations quantitatives de taux de convergence de certains schémas elliptiques, dits *monotones*, par différences finies (voir [35]). Il a aussi servi dans le cadre de l'homogénéisation pour déduire des taux de convergence [4, 36] (voir aussi [37]). Nous nous attendons à ce que le théorème 14 ait des applications similaires dans le cadre parabolique.

Nous donnons ensuite une première application de l'estimation  $W^{3,\varepsilon}$  parabolique en énonçant un résultat de régularité partielle pour les équations paraboliques associées à un opérateur uniformément elliptique de la forme :

$$\partial_t u + \mathcal{F}(D^2 u) = 0 \quad \text{sur } \Omega \subseteq \mathbb{R}^d \times (-1, \infty). \quad (57)$$

On rappelle que la dimension de Hausdorff (classique) est donnée par

$$\mathcal{H}(E) := \inf \{0 \leq s < +\infty : \text{pour tout } \delta > 0 \text{ il existe une collection } \{C_j\}, \text{diam}(C_j) \leq r_j \\ \text{telle que } E \subseteq \bigcup_{j=1}^{+\infty} C_j \text{ et } \sum_{j=1}^{+\infty} r_j^s < \delta\}. \quad (58)$$

**Théorème 15** (Régularité partielle parabolique, D.). *Supposons que l'opérateur  $\mathcal{F}$  satisfait (F1)-(F2). Soit  $u \in C(\Omega)$  une solution de viscosité de (57) sur un domaine  $\Omega \subseteq \mathbb{R}^{d+1}$ . Alors il existe une constante  $\varepsilon > 0$ , dépendant seulement de  $d, \lambda, \Lambda$ , et un sous-ensemble fermé  $\Sigma \subseteq \overline{\Omega}$  de dimension de Hausdorff au plus  $d + 1 - \varepsilon$ , tels que, pour tout  $0 < \alpha < 1$ , la solution  $u$  est de classe  $C^{2,\alpha}(\Omega \setminus \Sigma)$ .*

Il est important de noter que l'opérateur  $\mathcal{F}$  est uniformément elliptique mais aussi  $C^1$  et possède une différentielle uniformément continue. De plus, l'argument pour le résultat de régularité partielle est similaire à l'idée esquissée dans [2]. En effet, on applique un résultat de Wang [146], qui dit que toute solution de viscosité de (55) qui est suffisamment proche d'un polynôme quadratique doit être  $C^{2,\alpha}$ . Ce résultat est une généralisation d'un résultat de Savin dans le cas elliptique pour obtenir un résultat de régularité sur l'équation de Ginzburg-Landau [126]. Le théorème fournit l'existence de tels développements quadratiques sauf sur un ensemble de dimension de Hausdorff strictement inférieure.

### 2.2.2 Éléments de la preuve de l'estimation $W^{3,\varepsilon}$ parabolique

Afin d'expliquer la stratégie de preuve de l'estimation  $W^{3,\varepsilon}$  parabolique, nous allons tout d'abord énoncer l'estimation  $W^{2,\varepsilon}$  parabolique qui est l'élément-clé pour obtenir l'estimation  $W^{3,\varepsilon}$  parabolique puis nous expliquerons la stratégie pour prouver l'estimation  $W^{3,\varepsilon}$  parabolique. Nous nous contentons dans la présente section de donner l'énoncé de l'estimation  $W^{2,\varepsilon}$  parabolique, les éléments principaux de sa preuve étant proposés à la sous-section 0.2.2.3 qui suit.

Pour énoncer l'estimée, nous avons besoin d'introduire au préalable quelques définitions. Pour un domaine  $\Omega \subseteq \mathbb{R}^{d+1}$  et une fonction  $u \in \text{LSC}(\Omega)$ , on définit la quantité

$$\underline{\Theta}(x, t) = \underline{\Theta}(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ tel que } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. u(y, s) \geq u(x, t) + p \cdot (y - x) - A \left( \frac{1}{2} |x - y|^2 + (t - s) \right) \right\}.$$

De façon similaire, pour  $u \in \text{USC}(\Omega)$ ,

$$\overline{\Theta}(x, t) = \overline{\Theta}(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ tel que } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. u(y, s) \leq u(x, t) + p \cdot (y - x) + A \left( \frac{1}{2} |x - y|^2 + (t - s) \right) \right\},$$

et, pour  $u \in C(\Omega)$ ,

$$\Theta(x, t) = \Theta(u, \Omega)(x, t) := \max \{ \underline{\Theta}(u, \Omega)(x, t), \overline{\Theta}(u, \Omega)(x, t) \}.$$

La quantité  $\underline{\Theta}(x, t)$  est la courbure minimale d'un paraboloïde qui touche  $u$  par dessous au point  $(x, t)$ . Si  $u$  ne peut pas être touché par dessous au point  $(x, t)$  par aucun paraboloïde, alors  $\underline{\Theta}(x, t) = +\infty$ . Un énoncé similaire peut être donné pour  $\overline{\Theta}(x, t)$ , en touchant la fonction  $u$  par-dessus au point  $(x, t)$ . De plus, une fonction  $u$  est  $C^{1,1}$  sur un ensemble fermé  $\Gamma \subseteq \Omega$  si et seulement si  $u$  est touché par des paraboloïdes tangents par-dessus et par-dessous par rapport au domaine  $\Omega$  en tout point de  $\Gamma$ .

L'estimation clé est donnée par la proposition suivante. Elle contient l'information liée à l'inégalité parabolique ABP.

**Proposition 16** (Estimation  $W^{2,\varepsilon}$  parabolique). *Si  $u \in \text{LSC}(Q_1)$  et  $L \geq 0$  satisfont l'inégalité*

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -L \quad \text{in } Q_1,$$

*alors, pour tout  $\kappa > 0$ ,*

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u, Q_1)(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u| + L} \right)^{-\varepsilon}, \quad (59)$$

*où les constantes  $C$  et  $\varepsilon > 0$  sont universelles.*

Pour expliquer la preuve de l'estimation parabolique, nous allons partir du cas elliptique. La stratégie de la preuve de l'estimée  $W^{3,\varepsilon}$  elliptique donnée par la proposition 5 est la suivante :

**Etape 1 :** Etablir des estimées  $W^{2,\varepsilon}$  de mesure sur les quantités  $\underline{\Theta}(u, B_1)(x)$  et  $\overline{\Theta}(u, B_1)(x)$  elliptiques – qui donnent l'ouverture d'un paraboloïde qui touche par-dessous ou dessus la fonction  $u$  au point  $x$  sur l'ensemble  $B_1$  – pour  $u$  vérifiant des inéquations sur des opérateurs de Pucci.

**Etape 2 :** Etablir que pour  $u$  est de classe  $C^1$ ,

$$\Psi(u, B_1)(x) \leq \sum_{i=1}^d \Theta(u_{x_i}, B_1)(x). \quad (60)$$

où  $\Theta(u_{x_i}, B_1)(x) := \max\{\underline{\Theta}(u_{x_i}, B_1)(x), \overline{\Theta}(u_{x_i}, B_1)(x)\}$ .

**Etape 3 :** En utilisant les équation vérifiées par  $u_e = Du \cdot e$  pour tout vecteur unitaire  $e$ , on déduit l'estimée  $W^{3,\varepsilon}$ .

L'idée sous-jacente de la preuve du théorème 14 est similaire à celle du cas elliptique : on différencie l'équation pour obtenir le résultat à partir de l'estimation  $W^{2,\varepsilon}$  parabolique. Cependant, le passage au cas parabolique n'est pas immédiat et présente plusieurs difficultés, dues surtout au contrôle de la dérivée par rapport au temps de la solution. En effet, l'estimée intérieure  $C^{1,\alpha}$  parabolique de Harnack donne une régularité insuffisante en temps : si  $u$  est solution de viscosité de (55),  $u$  est de classe  $C^1$  en espace mais  $u$  est seulement  $C^{1/2}$  en temps et on ne peut donc rien dire a priori sur la régularité de la dérivée temporelle de  $u$ . Il faut donc trouver un autre moyen pour relier les quantités  $\Psi(u, B_1)$  et  $\Theta(u_{x_i}, B_1)$ ,  $1 \leq i \leq d$  correspondant à l'étape 2 de la preuve elliptique. Pour résoudre cette difficulté, on va :

- Utiliser une nouvelle fois l'EDP pour exprimer la dérivée temporelle  $\partial_t u$  en fonction de la non-linéarité  $\mathcal{F}$  et de  $D^2u$  (et éventuellement  $Du$  ainsi que les variables  $x$  et  $t$  pour des opérateurs plus généraux). **C'est différent du cas elliptique.**
- Montrer qu'en contrôlant toutes les dérivées spatiales, on dispose d'un contrôle suffisant sur la dérivée partielle  $\partial_t u$  en utilisant un argument par principe de comparaison en construisant une fonction barrière adéquate.

Nous pouvons maintenant décrire la stratégie de la preuve de l'estimée  $W^{3,\varepsilon}$  parabolique :

**Etape 1 :** Par différenciation en espace de l'équation (55) vérifiée par  $u$ , les fonctions gradients  $u_e = Du \cdot e$  satisfont les inégalités suivantes au sens de viscosité

$$\partial_t u_e + \mathcal{P}_{\lambda,\Lambda}^-(u_e) - \|g\|_{C^{0,1}(Q_1)} \leq 0 \leq \partial_t u_e + \mathcal{P}_{\lambda,\Lambda}^+(u_e) + \|g\|_{C^{0,1}(Q_1)}, \quad \text{sur } Q_1.$$

où  $\mathcal{P}_{\lambda,\Lambda}^-$  et  $\mathcal{P}_{\lambda,\Lambda}^+$  sont les opérateurs de Pucci donné par (1).

**Etape 2 :** On relie l'ensemble  $\{\Psi(u, Q_1) \leq C(1 + \|g\|_{C^{0,1}(Q_1)})\kappa\}$  et les  $d$  ensembles  $\{\Theta(u_{x_i}, Q_1) \leq \kappa\}$ ,  $1 \leq i \leq d$ , où  $C \geq 1$  est une constante universelle, pour  $u$  de classe  $C^1$  en espace et solution de (55). Plus précisément, on montre que, pour  $|\kappa| \geq 1$ ,

$$\bigcap_{i=1}^d \left\{ (x, t) \in Q_{1/2} \left(0, -\frac{1}{4}\right) : \Theta(u_{x_i}, Q_1)(x, t) \leq |\kappa| \right\} \subseteq \left\{ (x, t) \in Q_{1/2} \left(0, -\frac{1}{4}\right) : \Psi(u, Q_1)(x, t) \leq C(1 + \|g\|_{C^{0,1}(Q_1)})|\kappa| \right\}.$$

Cette inclusion signifie que le contrôle des dérivées en espace simultanément suivant toutes les directions implique le contrôle du comportement en temps de la solution. C'est ici que l'équation est utilisée afin de déterminer le paramètre  $b$  dans l'approximation quadratique parabolique. Cette étape est cruciale et remplace la seconde étape du cas elliptique.

**Etape 3 :** L'étape 2 fournit le lien entre  $\Psi(u, Q_1)(x, t)$  et les quantités  $\Theta(u_e, Q_1)(x, t)$ . Par l'étape 1, on peut appliquer l'estimée  $W^{2,\varepsilon}$  parabolique (proposition 16) aux fonctions  $u_e$  pour obtenir des estimations de mesure sur les quantités  $\underline{\Theta}[u_e]$  et  $\overline{\Theta}[u_e]$  suivant toutes les directions.

### 2.2.3 Preuve de l'estimation $W^{2,\varepsilon}$ parabolique

Au lieu de travailler avec les ensembles  $\{\underline{Q} \leq \kappa\}$ , nous allons introduire des nouveaux ensembles  $A_\kappa$  pour  $\kappa > 0$ . Pour cela, nous nous sommes inspirés du cadre elliptique introduit par Savin [126] dans le contexte de l'étude des ensembles de niveaux plats de l'équation de Ginzburg-Landau. Cette définition a été récemment reprise par Armstrong et Smart [5]. Dans le cas parabolique, on définit, pour tout  $\kappa > 0$ , l'ensemble

$$A_\kappa := \left\{ (x, t) \in Q_1 : \exists (y, s) \in B_1 \times (-1, t] \text{ tel que } u(x, t) - \inf_{Q_1} u + \kappa \left( \frac{1}{2} |x - y|^2 - (t - s) \right) \right. \\ \left. = \inf_{\substack{(z, \tau) \in Q_1, \\ \tau \leq t}} \left( u(z, \tau) - \inf_{Q_1} u + \kappa \left( \frac{1}{2} |z - y|^2 - (\tau - s) \right) \right) = 0 \right\}. \quad (61)$$

Il est important de noter que l'instant  $s$  intervient dans la définition de  $A_\kappa$  seulement pour permettre d'ajuster la valeur de l'infimum à zéro. De plus, la définition donnée ici de l'ensemble  $A_\kappa$  est adaptée au domaine  $Q_1$  mais elle peut être adaptée de façon naturelle en vue de traiter des domaines plus généraux.

**Lemme 17** (Estimation ABP parabolique). *Supposons que  $L > 0$  et  $u \in \text{LSC}(Q_1)$  satisfait*

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -L \text{ in } Q_1.$$

*Supposons que  $a > 0$  et  $V \subseteq \mathbb{R}^{d+1}$  est compact tel que, pour tout  $(y, s) \in V$ , il existe  $(x, t) \in Q_1$  tel que*

$$u(x, t) - \inf_{Q_1} u + \frac{a}{2} |x - y|^2 - a(t - s) = \inf_{\substack{(z, \tau) \in Q_1, \\ \tau \leq t}} [u(z, \tau) - \inf_{Q_1} u + \frac{a}{2} |z - y|^2 - a(\tau - s)] = 0. \quad (62)$$

*Soit  $W := \{(x, t) \in Q_1 : (2.3.5) \text{ est vérifiée pour } u \text{ pour un certain } (y, s) \in V\}$ . Alors*

$$|V| \leq \frac{1}{\lambda^d} \left( 1 + \frac{L}{a} + \Lambda d \right)^{d+1} |W|. \quad (63)$$

La mise à l'échelle parabolique pour un hamiltonien homogène nous dit que si  $u$  est solution d'une EDP avec un hamiltonien homogène  $H$ , alors la fonction  $v$  donnée par  $v(x, t) := \frac{1}{\lambda^2} u(\lambda x, \lambda^2 t)$  est également solution de l'EDP. D'autre part, ce scaling conduit aussi à introduire des ensembles géométriques adaptés. Pour  $(x, t) \in \mathbb{R}^{d+1}$ , on définit les *boules paraboliques* de centre  $(x, t)$ , d'ouverture  $\theta > 0$  et de hauteur  $h$  par :

$$G_{\theta, h}(x, t) := \{(y, s) : \theta |y - x|^2 \leq s - t \leq h\} \quad \text{et} \quad G_{\theta, h}^-(x, t) := \{(y, s) : \theta |y - x|^2 \leq t - s \leq h\}.$$

La stratégie pour obtenir l'estimation  $W^{2,\varepsilon}$  donnée par la proposition 16 est la suivante :

**Etape 1 :** Nous découpons le cylindre  $Q_{1/2}(0, -\frac{1}{4})$  par une famille finie de cylindres paraboliques suffisamment petits (voir Figure 5) de sorte que chaque cylindre de rayon  $R$  peut être inclus dans une boule parabolique  $G_{\theta, H}(x_i, t_i)$  possédant les caractéristiques suivantes :

- la hauteur  $H$  de la boule parabolique est comparable à  $R$ ;
- le sommet  $(x_i, t_i)$  de la boule parabolique appartient à  $A_{\kappa_0}$ .

Il suffit alors d'obtenir l'estimation suivante de décroissance en mesure

$$|G_{\theta, H_0}^-(x_0, t_0) \setminus A_\kappa| \leq C \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon}, \quad (64)$$

qui va être l'objet des étapes 2-3.

**Etape 2 :** Nous montrons une estimation en mesure valable à toute échelle  $h > 0$  pour toute boule parabolique  $G_{\theta,h}(x, t)$  de sommet  $(x, t)$  et hauteur  $h > 0$  vérifiant certaines conditions. Plus précisément, il existe deux constantes universelles  $M \geq 1$  et  $\sigma > 0$  telle que pour tout  $(x, t) \in A_\kappa$  et toute hauteur  $h > 0$  satisfaisant

$$G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h\} \cap G_{\theta,h_0}(x_0, t_0) \cap A_\kappa \neq \emptyset$$

alors nous avons

$$|G_{\theta,h}(x, t) \cap A_{M\kappa} \cap G_{\theta,h_0}^-(x_0, t_0)| \geq \sigma |G_{\theta,h}(x, t)|.$$

Cette estimation consiste à montrer qu'étant donné un point  $(x_0, t_0) \in A_\kappa$ , il est possible de construire un ensemble, de mesure strictement positive, contenu dans la boule parabolique  $G_{\theta,h}(x, t)$  et dans  $A_{M\kappa}$ . Pour cela, nous mettons en oeuvre d'une part le lemme 2.3.3 (inégalité ABP parabolique) et d'autre part un argument par fonction barrière mettant en jeu la solution fondamentale de l'équation de la chaleur.

**Etape 3 :** Afin de déduire l'estimation (64), nous recollons ces estimations valables à toute hauteur  $h > 0$  par un lemme de recouvrement de Vitali adapté aux boules paraboliques<sup>1</sup>.

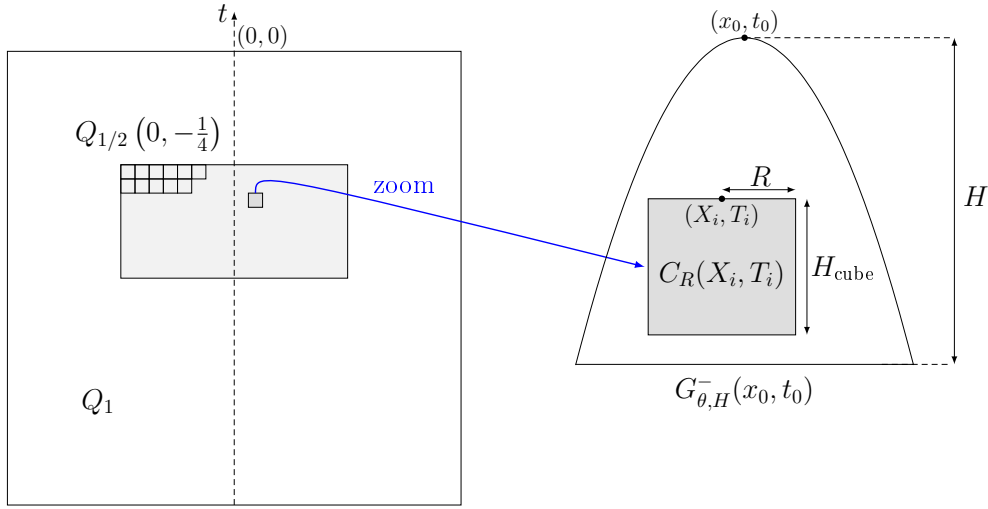


Figure 5 – Estimation  $W^{2,\varepsilon}$  parabolique : description géométrique de l'argument par recouvrement : le cube est recouvert par une collection finie de cylindres paraboliques disjoints de dimensions identiques. Il est ensuite possible de trouver une valeur  $\kappa_0$  pas trop grande de sorte que chaque cube peut être inclus dans une boule parabolique dont le sommet appartient à  $A_{\kappa_0}$ .

#### 2.2.4 Preuve du résultat de régularité partielle parabolique

Donnons quelques idées de la preuve du théorème de régularité partielle. En plus de l'estimée  $W^{3,\varepsilon}$  parabolique, nous nous appuyons sur un résultat établi récemment par Yu Wang dont l'on va maintenant rappeler l'énoncé.

**Proposition 18** (Y. Wang, [146]). *Supposons en plus de (F1)–(F2) que  $\mathcal{F}(0) = 0$  et  $g(0,0) = 0$ . Supposons que  $0 < \alpha < 1$  et  $u \in C(Q_1)$  est une solution de (55) sur  $Q_1$ . Alors il existe une constante*

1. Il existe dans la littérature trois grands types d'argument de recouvrement : le lemme de Besicovitch, la décomposition de Calderón-Zygmund et le lemme de Vitali. La décomposition de Calderón-Zygmund est l'argument de recouvrement utilisé par Caffarelli dans [31] donnant la régularité elliptique pour les solutions de viscosité mais se révèle extrêmement technique dans le cas parabolique. Une version du lemme de Vitali, mettant ici en oeuvre pour les boules paraboliques, se révèle très pratique.

$\delta_0 > 0$  dépendant seulement des constantes d'ellipticité  $\lambda$  and  $\Lambda$ , la dimension  $d$ , le module de continuité  $\omega$ , et  $\alpha$ , telle que

$$\sup_{Q_1} |u| \leq \delta_0 \implies u \in C^{2,\alpha}(Q_{1/2}).$$

Ce résultat est la version parabolique d'un résultat elliptique établi par Savin [126, Théorème 1.1]. Sa motivation était l'étude des ensembles de niveaux plats dans les modèles de transition de phase de Ginzburg-Landau proposés dans [127].

La preuve du théorème de régularité partielle parabolique est strictement parallèle à la preuve du théorème de régularité partielle elliptique.

**Etape 1 :** Nous considérons l'ensemble  $\Sigma$  des points singuliers pour lesquels  $u$  n'admet pas de développement  $C^{2,\alpha}$ . Le scaling parabolique nous conduit à considérer un recouvrement par des cylindres paraboliques de taille  $r$  en espace et  $r^2$  en temps. Nous introduisons la dimension de Hausdorff *parabolique* définie par :

$$\mathcal{H}_{\text{par}}(E) := \inf \left\{ 0 \leq s < +\infty : \text{pour tout } \delta > 0 \text{ il existe une collection } \{Q_{r_j}(x_j, t_j)\} \right. \\ \left. \text{de cylindres tels que } E \subseteq \bigcup_{j=1}^{+\infty} Q_{r_j}(x_j, t_j) \text{ et } \sum_{j=1}^{+\infty} r_j^s < \delta \right\}.$$

**Etape 2 :** Nous montrons que  $\mathcal{H}_{\text{par}}(\Sigma) \leq d + 2 - \varepsilon$ . Nous procédons de la façon suivante : en appliquant la proposition 18 sur la solution  $u$  regardée avec un changement d'échelle parabolique, nous montrons que si nous sommes capables de trouver à l'intérieur d'un cylindre parabolique  $Q_r(y_0, s_0)$  de taille petite arbitraire un point  $(z, s)$  telle que la taille du terme cubique n'est pas "trop grand", alors  $u$  est de classe  $C^{2,\alpha}$  dans un cylindre  $Q_r(y_0, s_0 - r^2)$  donc de même taille que le premier cylindre mais légèrement shifté backward en temps (le shift étant égal à  $r^2$ ). Plus précisément, nous avons l'implication :

$$\Psi(u, Q_{3/4})(z, s) \leq \delta_\alpha r^{-1} \text{ sur } Q_r(y_0, s_0) \implies u \text{ est } C^{2,\alpha} \text{ sur un cylindre } Q_r(y_0, s_0 - r^2).$$

L'énoncé précis est fourni par le lemme 2.2.5 au chapitre 2.

**Etape 3 :** Nous montrons que  $\mathcal{H}(\Sigma) \leq d + 1 - \frac{\varepsilon}{2}$  en utilisant la relation liant les dimensions de Hausdorff parabolique et usuelle – donnée par (58) – et parabolique par la relation

$$2\mathcal{H}(E) - d \leq \mathcal{H}_{\text{par}}(E) \leq \mathcal{H}(E) + 1.$$

Les deux cas limites sont possibles. En particulier, nous avons  $\mathcal{H}_{\text{par}}(\mathbb{R}^{d+1}) = d + 2$ .

## 2.3 Estimation d'erreurs pour des schémas explicites et implicites pour des équations elliptiques ou paraboliques non-linéaires

Cette partie reprend les résultats présentés dans les chapitres 3 et 4. Nous nous intéressons dans cette partie à l'étude du taux de convergence pour des schémas associés à des équations elliptiques et paraboliques non-linéaires munies d'une condition d'uniforme ellipticité. Parmi la grande variété de schémas possibles, nous allons envisager deux types de schémas :

- Les schémas donnés par un principe de programmation dynamique et plus particulièrement ceux proposés par Kohn et Serfaty via le jeu de Mark et Helen. Comme rappelé dans la première partie, ce jeu fournit une interprétation de type contrôle déterministe pour les équations elliptiques et paraboliques non-linéaires générales. Nous étudierons le cas elliptique.
- Les schémas par différences finies sur une grille  $\Gamma_\varepsilon := \varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z}$  en espace et en temps pour les problèmes paraboliques.

Une distinction importante entre les types d'équations est l'ordre de l'équation. Les méthodes sont en effet très différentes entre les équations du premier ordre (Hamilton-Jacobi) et celles du second ordre (elliptiques) non-linéaires. Pour les équations du premier ordre, si la méthode des caractéristiques est utilisable, des formules exactes peuvent être mis en oeuvre (par exemple la formule de Hopf-Lax) et il existe un lien avec les équations de conservation du premier ordre [61]. Les équations du second ordre possèdent plus de points communs avec les équations elliptiques sous forme divergence, mais à cause de la non-linéarité ou de la dégénérescence, la plupart des outils provenant du cas sous forme divergence (e.g. les éléments finis, ou les solveurs multi-grilles) n'ont pas été appliqués avec succès.

Pour les EDP uniformément elliptiques, les schémas monotones ne sont pas nécessaires pour la convergence (c'est le cas par exemple pour les méthodes d'éléments finis d'ordre élevés qui ne sont pas monotones). Mais pour des équations non-linéaires ou dégénérées, la seule preuve de convergence, qui a été fournie par Oberman [120] dans le cadre des schémas par différences finies, repose sur des schémas monotones.

Pour des schémas issus du principe de programmation dynamique, Soravia [132] a étudié le taux de convergence d'un schéma donné par des jeux différentiels vers la solution d'une équation d'Isaacs. Tataru s'est intéressé à des équations de programmation dynamique dans un Hilbert (voir [139]). Des schémas d'approximation pour des équations du premier ordre ont été traités par Crandall et Lions (voir [43]).

Oberman a également étudié des schémas par différences finies pour le  $p$ -laplacien et l'équation de Monge-Ampère avec Froese (voir [70, 71]), ainsi que les ensembles de niveaux pour le mouvement par courbure moyenne [119]. Par d'autres moyens, Crandall Lions [49] ont donné des schémas explicites pour la version parabolique du mouvement par courbure moyenne. Krylov a obtenu des taux de convergence pour des schémas paraboliques associés à des équations de Bellman avec des coefficients présentant une régularité Lipschitz (voir [95], [56] et [111]).

La convergence d'approximations monotones et consistantes d'EDP non linéaires du second ordre a été établie pour la première fois par Barles et Souganidis (voir [20]). Kuo et Trudinger ([98, 99]) ont plus tard étudiés l'existence de d'approximations monotones et consistantes pour les équations non-linéaires et la régularité des solutions approchées. Ils ont montré que si  $\mathcal{F}$  est uniformément elliptique, il existe alors un schéma  $F_h$  monotone par différences finies qui est consistant avec  $\mathcal{F}$ , et que les solutions approchées  $u_h$  sont de classe  $C^{0,\eta}$ . Cependant estimer l'erreur restait un problème ouvert.

Une première méthode d'obtention d'un taux de convergence a été proposée par Caffarelli et Souganidis (voir [35]). L'idée consiste essentiellement à adapter la preuve classique de l'unicité des solutions de viscosité donnée dans [34, Théorème 5.3]. Les auteurs combinent l'inégalité ABP et la régularisation par inf-sup convolution. On peut également signaler la mise en Schéma pour le mouvement par courbure moyenne, lien avec les jeux [40].

Les premières estimations d'erreur pour des schémas d'approximation ont été établis par Krylov [93, 94] pour des équations convexes ou bien concaves, mais qui pouvaient être cependant dégénérés. L'idée de Krylov a été d'utiliser des méthodes de contrôle stochastique qui s'appliquent dans le cas convexe ou concave, mais cependant pas dans le cadre général. Barles et Jakobsen ont amélioré dans [15] et [16] les estimations d'erreur de Krylov pour des équations convexes ou concaves. Dans [95], Krylov a amélioré l'estimation d'erreur pour être d'ordre  $\varepsilon^{1/2}$  (où  $\varepsilon$  est le pas de discrétisation), mais toujours dans le cas convexe ou concave. De plus, Jakobsen [84, 85] et Bonnans, Maroso et Zidani [25] ont ainsi pu établi des estimations d'erreur pour des équations particulières ou pour certaines dimensions.

Nous allons présenter l'approche générale suivie pour nos schémas. Il s'est avéré que la stratégie poursuivie a pu être appliquée pour des schémas très différents qui pouvaient être associés elliptiques mais aussi et surtout paraboliques (avec bien sûr des variantes). Nous nous proposons donc d'étudier le taux de convergence dans ces deux cadres elliptique et parabolique pour des schémas issus d'une part du principe de programmation dynamique et d'autre part des schémas par différences finies. Signalons tout de suite que les schémas de principe de programmation dynamique ne peuvent être directement implémentés par la définition même de l'opérateur de principe de programmation dynamique, bien que

l'on peut bien sûr imaginer de discrétiser cet opérateur. Le taux obtenu est donc abstrait dans notre cas.

Nous allons expliquer les hypothèses requises pour étudier un taux de convergence pour de telles équations :

- La non-linéarité  $\mathcal{F}$  satisfait l'hypothèse (F1) d'uniforme ellipticité. Pour obtenir un taux, l'hypothèse d'ellipticité simple (E) ne fournit pas d'information suffisante et il est alors nécessaire de considérer l'hypothèse d'uniforme ellipticité (F1). Cette condition d'uniforme ellipticité correspond à une propriété de stabilité puisqu'elle contrôle de manière Lipschitz la taille d'une perturbation autour d'une matrice. L'équation possédant un principe de comparaison, cela permet de déduire des inégalités ABP portant sur la caractérisation de l'enveloppe convexe (resp. monotone) par l'utilisation d'arguments par fonctions barrières dans le cadre elliptique (resp. parabolique).
- Le schéma d'approximation  $F_\varepsilon$  doit être *consistant* par rapport à la non-linéarité  $\mathcal{F}$ . Cela signifie qu'il doit être une bonne approximation de la non-linéarité  $\mathcal{F}$ , c'est-à-dire qu'il doit calculer correctement l'action de la non-linéarité pour des fonctions régulières, plus précisément de classe  $C^3$  car nos équations sont du second ordre.
- Le schéma  $F_\varepsilon$  doit satisfaire un principe de comparaison. En utilisant conjointement l'hypothèse de consistance et des arguments par fonction barrière sur le schéma, nous allons être capable d'élaborer des inégalités de type ABP sur le schéma d'approximation. Il est ici naturel qu'un terme d'erreur dû au schéma va intervenir. Nous reviendrons sur ce point.

On souhaite montrer un résultat de la forme suivante. Si  $u$  et  $u^\varepsilon$  désignent respectivement la solution de l'EDP et la solution du schéma, alors il existe des constantes  $C$  et  $\eta$  dépendant de la dimension ambiante, des constantes d'ellipticité, du domaine  $\Omega$  et de la donnée au bord (et éventuellement de la donnée initiale) telles que

$$\sup_{\overline{\Omega}} |u - u_\varepsilon| \leq C\varepsilon^{\eta*}.$$

L'étude du taux est effectuée suivant deux étapes :

- *Etude au voisinage du bord.* Cette étape se ramène à étudier la consistance du schéma par rapport à la condition au bord et à évaluer l'erreur commise dans un voisinage du bord par le schéma. Cela est effectué en mobilisant des arguments par fonction barrière et les principe de comparaison associé au schéma. Nous utilisons aussi, sous l'hypothèse (F1) d'uniforme ellipticité, la régularité  $C^{0,\alpha}(\overline{U})$  de la solution de viscosité de l'EDP jusqu'au bord et en particulier la consistance avec la condition au bord  $\partial U$ .
- *Etude à l'intérieur du domaine.* Le raisonnement est par l'absurde : on suppose que l'erreur à l'intérieur du domaine est trop grande pour une certaine puissance  $\varepsilon^\eta$  avec  $\eta < \eta_*$  et on cherche alors à aboutir une contradiction. Nous mobiliserons les deux outils suivants de la théorie de la régularité :
  - ◊ L'estimation  $W^{3,\varepsilon}$  elliptique (resp. parabolique) portant sur la mesure de Lebesgue des points admettant un développement de Taylor du second ordre avec un contrôle sur le terme cubique pour pallier le défaut de régularité  $C^2$ .
  - ◊ L'inégalité ABP elliptique (resp. parabolique) en étudiant soigneusement l'enveloppe convexe (resp. monotone) d'une certaine quantité associée à l'erreur commise par la solution du schéma qui approche la solution  $u$  de l'EDP.

### 2.3.1 Schémas elliptiques issus du principe de programmation dynamique

Nous nous sommes intéressés à l'équation elliptique

$$\begin{cases} \mathcal{F}(D^2u) + \xi u = 0, & \text{sur } U \subseteq \mathbb{R}^d, \\ u = g, & \text{sur } \partial U, \end{cases} \quad (65)$$



où  $\xi > 0$  et  $\mathcal{F}$  satisfait (F1) et  $U$  est un domaine régulier.

Kohn et Serfaty [88] ont proposé une interprétation par contrôle déterministe d'une large classe d'équations elliptiques et paraboliques non-linéaires par une famille de jeux répétés à deux personnes. Comme cela a été expliqué à la section 0.1.3 de la présente introduction, le jeu construit fournit un schéma d'approximation de la solution de viscosité. Une question naturelle, déjà soulevée dans [88, Section 4.1], est d'étudier le taux de convergence de la solution donnée par le jeu, en imposant éventuellement des conditions sur  $\mathcal{F}$ .

On répond à la question posée pour le problème de Dirichlet elliptique en résolvant (65) pour une non-linéarité satisfaisant dans un domaine régulier muni de la condition au bord  $u = g$  sur  $\partial U$ . Par souci de simplicité, nous allons nous concentrer sur la situation où la non-linéarité  $\mathcal{F}$  dépend seulement de la matrice hessienne. En particulier, l'EDP possède un principe de comparaison et la constante  $\xi$  qui apparaît dans (65) et qui joue le rôle d'un taux d'intérêt dans le jeu n'a pas de condition supplémentaire puisque  $\mathcal{F}$  est indépendant de  $u$ . Dans ce cas, la fonction valeur  $u_\varepsilon$  vérifie l'équation de programmation dynamique

$$u_\varepsilon(x) = S_\varepsilon[x, u_\varepsilon] \quad (66)$$

où  $S_\varepsilon$  est l'opérateur défini pour tout  $x \in U$ ,  $z \in \mathbb{R}$ , et toute fonction continue  $\phi : U \rightarrow \mathbb{R}$ , par

$$S_\varepsilon[x, \phi] = \sup_{p, \Gamma} \inf_{\Delta x} \left[ e^{-\xi \varepsilon^2} \hat{\phi}(x + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \right], \quad (67)$$

avec les conventions usuelles que  $p$ ,  $\Gamma$  et  $\Delta x$  vérifient les contraintes

$$\|p\| \leq \varepsilon^{-\beta} \quad \text{et} \quad \|\Gamma\| \leq \varepsilon^{-\gamma} \quad (68)$$

et

$$\|\Delta x\| \leq \varepsilon^{1-\alpha}. \quad (69)$$

ainsi que la convention que, pour  $\phi \in C(U)$ ,  $\hat{\phi}$  représente la fonction définie par

$$\hat{\phi}(x) = \begin{cases} \phi(x), & \text{if } x \in U, \\ g(x), & \text{if } x \notin U. \end{cases}$$

Comme le calcul de  $S_\varepsilon[x, \phi]$  est entièrement déterminé par  $\phi$  sur la boule  $B(x, \varepsilon^{1-\alpha})$ , nous allons définir des notions d'adhérence et de frontière approchées adaptées au schéma en posant :

$$\text{Adh}_\varepsilon H := \{x \in \mathbb{R}^d : \text{dist}(x, H) \leq \varepsilon^{1-\alpha}\}$$

et la frontière approchée de  $H$  par

$$\partial_\varepsilon H := \text{Adh}_\varepsilon H \setminus H. \quad (70)$$

On rappelle que l'opérateur  $S_\varepsilon$  vérifie les trois propriétés suivantes :

- Pour tout  $\phi \in C(\overline{\Omega})$ ,  $S_0[x, \phi] = \phi(x)$ .
- $S_\varepsilon$  est monotone, i.e. si  $\phi_1 \leq \phi_2$ , alors  $S_\varepsilon[x, \phi_1] \leq S_\varepsilon[x, \phi_2]$ .
- Pour tout  $\phi \in C(\overline{\Omega})$  et  $c \in \mathbb{R}$ ,  $S_\varepsilon[x, c + \phi] = e^{-\xi \varepsilon^2} c + S_\varepsilon[x, \phi]$ .

On définit maintenant l'opérateur  $F_\varepsilon$  remis à l'échelle défini pour  $x \in U$  et  $u \in L^\infty(U)$  par

$$F_\varepsilon[u](x) := \frac{1}{\varepsilon^2} (u(x) - S_\varepsilon[x, u]), \quad (71)$$

où l'opérateur  $S_\varepsilon$  est défini par (67). Il correspond au générateur, à  $\varepsilon$  fixé, associé à l'opérateur  $S_\varepsilon$  de programmation dynamique (on obtient le générateur infinitésimal en faisant tendre  $\varepsilon$  vers 0). La solution  $u^\varepsilon$  du schéma (66) est également solution de

$$F_\varepsilon[u^\varepsilon] = 0 \quad \text{sur } U. \quad (72)$$

Nous donnons maintenant les hypothèses utilisées sur la non-linéarité  $\mathcal{F}$  et le schéma d'approximation  $F_\varepsilon$ . Nous allons supposer que  $\mathcal{F}$  satisfait (F1) et également

(F3)  $\mathcal{F}(0) = 0$ ,

cette dernière hypothèse étant uniquement simplificatrice dans la conduite des calculs. Pour étudier le taux de convergence de ces schémas elliptiques, nous allons nous appuyer les deux propriétés suivantes de l'opérateur  $F_\varepsilon$  :

(S1) (**Principe de comparaison local pour  $F_\varepsilon$** ) Si  $u, -v \in \text{LSC}(U) \cap L^\infty(U)$  satisfont

$$F_\varepsilon[u] \leq F_\varepsilon[v] \quad \text{sur } H \subseteq U,$$

alors

$$\sup_H (u - v) \leq e^{-\xi\varepsilon^2} \max_{\partial_\varepsilon H} (u - v)$$

où  $\partial_\varepsilon H$  est défini par (70).

(S2) (**Consistance de  $F_\varepsilon$** ) Il existe une constante universelle  $C > 0$  telle que pour tout  $x$  et toute fonction régulière  $\phi$  défini au voisinage de  $x$ ,  $F_\varepsilon$  étant défini par (71), on a

$$|F_\varepsilon[x, \phi] - (\mathcal{F}(D^2\phi(x)) + \xi\phi(x))| \leq C \left( 1 + \sup_{B(x, \varepsilon^{1-\alpha})} \|D^3\phi\| \right) \varepsilon^\nu$$

avec  $\nu := \min\{\alpha, 1 - 3\alpha\}$ .

Nous énonçons le taux de convergence obtenu pour les schémas.

**Théorème 19** (Taux de convergence elliptique pour le jeu de Mark et Helen, D.). *Soit  $U$  un domaine borné de classe  $C^2$  et  $g \in C^{0,1}(\partial U)$ . Supposons que  $\mathcal{F}$  et  $F_\varepsilon$  satisfont respectivement (F1)-(F3) et (S1)-(S2) et que  $u$  et  $u_\varepsilon$  sont respectivement la solution de viscosité de (65) et de l'équation de programmation dynamique (66). Alors il existe des constantes  $\eta$  et  $C$  dépendant de  $\lambda, \Lambda, d$  et  $\|g\|_{C^{0,1}(\partial U)}$  telles que*

$$\sup_U |u - u_\varepsilon| \leq C\varepsilon^\eta.$$

### 2.3.2 Quelques éléments de preuve du taux

Nous allons donner seulement énoncer les éléments principaux de la preuve du taux elliptique car sa stratégie suit les mêmes étapes que celle de la preuve du taux parabolique. Comme cette dernière est la plus délicate, nous allons présenter soigneusement les difficultés rencontrées dans ce cadre.

Comme annoncé, la première étape consiste à étudier le taux de convergence au voisinage du bord. Cela nous conduit à analyser soigneusement le jeu afin d'obtenir des d'informations concernant la régularité de la solution  $u^\varepsilon$  du schéma associé au jeu ainsi que les propriétés (S1) et (S2) mentionnées ci-dessus. Dans un second temps, nous présenterons l'estimation  $W^{3,\varepsilon}$  elliptique ainsi que l'analyse par enveloppe convexe portant sur une quantité introduite pour étudier l'erreur commise par le schéma pour approximer la solution de (65). Tout d'abord, les propriétés de la solution du schéma sont rassemblées dans la proposition suivante.

**Proposition 20.** *Soit  $\varepsilon > 0$  suffisamment petit tel que la solution  $u_\varepsilon$  la solution du schéma (72) existe. Alors  $u_\varepsilon$  est continue sur  $U$  et bornée indépendamment de  $\varepsilon$ . De plus, si  $g \in C^{0,1}(\partial U)$ , alors il existe des constantes  $\eta \in (0, 1)$  et  $C > 0$  dépendant seulement de  $\lambda, \Lambda, d$  telles que, pour tout  $x, y \in U$ , nous avons*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C (1 + \|g\|_{C^{0,1}(\partial U)}) \rho_\varepsilon(x, y).$$

avec  $\rho_\varepsilon(x, y) := \max\{d(x, y), \varepsilon^{1-\alpha}\}$ .

La propriété (S1) est un principe de comparaison associé à l'opérateur discret. Elle permet d'introduire une notion de sous-solution et sur-solution associées au schéma  $F_\varepsilon$  et admet une version globale donnée par :

(S1G) **(Principe de comparaison global pour  $F_\varepsilon$ )** Supposons que  $u, -v \in \text{LSC}(U) \cap L^\infty(U)$  satisfont

$$F_\varepsilon[u] \leq F_\varepsilon[v] \quad \text{sur } U,$$

alors

$$\sup_U (u - v) \leq 0.$$

Nous allons maintenant présenter les outils pour étudier le taux de convergence à l'intérieur du domaine dans le cadre elliptique. Une difficulté technique à surmonter est que la solution de viscosité de l'équation n'est pas de classe  $C^2$  en général (sauf si elle est convexe ou concave). Pour la surmonter, nous utilisons l'estimée  $W^{3,\varepsilon}$  associée à l'équation (65). Cette estimation donne des développements de Taylor au second ordre pour les solutions à coefficients constants sur des ensembles de grande mesure de Lebesgue. Nous renvoyons à la section 0.1.4 pour davantage de détails sur la régularité elliptique de l'équation. Par souci d'autonomie de la présente partie, nous rappelons la définition essentielle. Nous définissons, pour  $u : U \rightarrow \mathbb{R}$ , la quantité

$$\Psi(u, U)(x) := \inf \left\{ A \geq 0 : \exists (p, M) \in \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } \forall y \in U, \right. \\ \left. |u(y) - u(x) + p \cdot (x - y) + \frac{1}{2}(x - y) \cdot M(x - y)| \leq \frac{1}{6}A|x - y|^3 \right\}.$$

**Proposition 21** (Estimation  $W^{3,\varepsilon}$  elliptique). *Supposons que  $\mathcal{F}$  est uniformément elliptique et  $u \in C(B_1)$  est une solution de (65) dans  $Q_1$ . Alors il existe des constantes universelles  $C, \varepsilon > 0$  telles que, pour tout  $\kappa > 0$ ,*

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{(1 + |\xi|) \sup_{B_1} |u| + |\mathcal{F}(0)|} \right)^{-\varepsilon}.$$

La proposition suivante est un principe de comparaison sur des variables doubles. Son énoncé et sa preuve sont analogues à [4, Lemma 6.1]. Elle donne une borne inférieure sur la mesure de Lebesgue dans  $\mathbb{R}^d \times \mathbb{R}^d$  de l'ensemble des points sur lesquels nous pouvons toucher la différence d'une sous-solution et d'une sur-solution par des plans, après avoir doublé les variables et ajouté le terme habituel de pénalisation quadratique.

**Proposition 22.** *Soit  $U \subseteq \mathbb{R}^d$ . Supposons que  $u \in C(U)$  et  $v : U \rightarrow \mathbb{R}$  satisfait*

$$\mathcal{P}_{1,\Lambda}^-(D^2 u) \leq 1 \quad \text{sur } U$$

et

$$F_\varepsilon[v] \geq -1 \quad \text{sur } U.$$

Fixons  $0 < \delta \leq 1$  et définissons, pour tout  $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ , une fonction  $\Phi_{p,q} : U \times U \rightarrow \mathbb{R}$  par

$$\Phi_{p,q}(x, y) := u(x) - v(y) - \frac{1}{2\delta}|x - y|^2 - p \cdot x - q \cdot y.$$

Supposons que  $V \subseteq \mathbb{R}^{2d}$  et  $W \subseteq \mathbb{R}^{2d}$  sont tels que  $\overline{V} \subseteq U \times U$  et, pour tout  $(p, q) \in W$ , il existe  $(x, y) \in V$  tels que

$$\Phi_{p,q}(x, y) = \sup \{ \Phi_{p,q}(\xi, \eta) : (\xi, \eta) \in U \times U \}.$$

Alors il existe une constante  $C > 1$  dépendant seulement de  $d, \Lambda$  et  $\text{diam } U$  telle que pour tout  $\rho > 0$ ,

$$|W| \leq C\delta^{-2d} \left( 1 + \frac{\varepsilon^{1-\alpha}}{\rho} \right)^{2d} |V + B_\rho \times B_\rho|.$$

### 2.3.3 Schémas par différences finies pour des équations paraboliques

Les résultats obtenus dans cette partie sont issus d'un travail en collaboration avec Scott Armstrong. Nous nous sommes intéressés à l'équation parabolique non-linéaire

$$\partial_t u + \mathcal{F}(D^2 u) = 0 \quad \text{sur } \mathbb{R}^d \times (0, \infty) \quad (73)$$

munie de la condition initiale

$$u = g \quad \text{sur } \mathbb{R}^d \times \{0\} \quad (74)$$

avec  $g$  régulière à support compact. Le schéma implicite en temps est donné par

$$\begin{cases} \hat{\partial}_t^\varepsilon v(x, t) + F_\varepsilon[v(\cdot, t)](x) = 0, & \text{sur } \Gamma_\varepsilon \cap \{(x, t) : t \geq \varepsilon^2\}, \\ v(\cdot) = g, & \text{sur } \partial_p \Gamma_\varepsilon, \end{cases} \quad (75)$$

où l'opérateur de dérivation temporelle discrète  $\hat{\partial}_t^\varepsilon$  s'écrit

$$\hat{\partial}_t^\varepsilon v(x, t) := \frac{1}{\varepsilon^2} (v(x, t) - v(x, t - \varepsilon^2))$$

et  $F_\varepsilon$  est un schéma par différences finies qui agit sur des fonctions spatiales. Nous notons  $\mathcal{E}$  l'ensemble des fonctions  $\phi : \varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z} \rightarrow \mathbb{R}$ . Les hypothèses formulées sur le schéma sont les suivantes :

(S1) *Monotonie* : Pour toutes fonctions  $\phi, \xi \in \mathcal{E}$  et  $z \in \varepsilon\mathbb{Z}^d$ , la condition

$$\phi(z) - \xi(z) > \sup_{\varepsilon\mathbb{Z}^d \setminus \{z\}} (\phi - \xi)$$

implique l'inégalité

$$F_\varepsilon[\phi](z) > F_\varepsilon[\xi](z).$$

(S2) *Consistance* : Il existe une constante  $K_0 > 1$  telle que, pour toute fonction  $\phi \in C^3(\mathbb{R}^d)$ ,

$$\sup_{z \in \varepsilon\mathbb{Z}^d} |\mathcal{F}(D^2 \phi(z)) - F_\varepsilon[\phi](z)| \leq K_0 \varepsilon \sup_{x \in \mathbb{R}^d} |D^3 \phi(x)|.$$

(S3) *Continuité et localité* : Il existe des constantes  $R_0 > 1$  et  $K_1 > 1$  telles que, pour tout  $\xi \in \mathcal{E}$ ,  $\phi \in C^2(\mathbb{R}^d)$  et  $z \in \varepsilon\mathbb{Z}^d$ ,

$$|F_\varepsilon[\xi + \phi](z) - F_\varepsilon[\xi](z)| \leq K_1 \sup_{x \in B_{R_0 \varepsilon}(z)} |D^2 \phi(x)|.$$

Nous pouvons maintenant énoncer le théorème de convergence pour des schémas paraboliques implicites en temps que nous avons obtenu.

**Théorème 23** (taux de convergence pour des schémas paraboliques, Armstrong et D.). *Soit  $\Omega = \mathbb{R}^d \times (0, \infty)$  et  $\Gamma_\varepsilon := \Omega \cap (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z})$ . Soit  $\mathcal{F}$  et  $F_\varepsilon$  satisfaisant respectivement (F1) et (F3) et (S1)–(S3),  $g \in C^{0,1}(\mathbb{R}^d)$  à support compact. Supposons que  $u : \Omega \rightarrow \mathbb{R}$  et  $v : \Gamma_\varepsilon \rightarrow \mathbb{R}$  sont solutions respectivement de (73)–(74) et (75). Alors pour tout  $T > 0$ , il existe des constantes  $\eta \in (0, 1)$  et  $C > 0$  dépendant des constantes d'ellipticité  $\lambda, \Lambda$ , la dimension  $d$  et de  $\|g\|_{C^{0,1}(\mathbb{R}^d)}$  telles que*

$$\max_{\Gamma_\varepsilon \cap \{(x, t) \in \mathbb{R}^d \times [0, T]\}} |(u - v)| \leq C \varepsilon^\eta.$$

Nous allons présenter rapidement la stratégie de la preuve.

Nous avons besoin de définir l'enveloppe monotone (parabolique) qui est la version parabolique de l'enveloppe convexe. Si  $\Omega \subseteq \mathbb{R}^{d+1}$  est ouvert, alors l'enveloppe monotone  $\Gamma_u$  d'une fonction semi-continue inférieurement  $u : \Omega \rightarrow \mathbb{R}$  est la plus grande fonction  $v : \Omega \rightarrow \mathbb{R}$  étant en-dessous de  $u$  qui est décroissante

par rapport à  $t$  et convexe par rapport à  $x$ . L'enveloppe monotone d'une fonction  $u \in \text{LSC}(\Omega)$  admet la représentation suivante

$$\Gamma_u(x, t) := \sup_{p \in \mathbb{R}^d} \inf_{(y, s) \in \Omega, s \leq t} (u(y, s) + p \cdot (x - y)). \quad (76)$$

Si  $(x, t) \in \{u = \Gamma_u\}$ , on peut définir un vecteur  $(p, h) \in \mathbb{R}^{d+1}$  associé à  $(x, t)$  par la relation

$$h = u(x, t) - p \cdot x = \inf_{(y, s) \in \Omega, s \leq t} \{u(y, s) - p \cdot y\}. \quad (77)$$

La proposition suivante est un principe de comparaison sur des variables doubles. Son énoncé et sa preuve sont analogues à [4, Lemma 6.1]. Elle donne une borne inférieure sur la mesure de Lebesgue dans  $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$  de l'ensemble des points sur lesquels nous pouvons toucher la différence d'une sous-solution et d'une sur-solution par des plans, après avoir doublé les variables et ajouté le terme habituel de pénalisation quadratique. La preuve s'appuie sur deux inégalités ABP paraboliques données par les lemmes 25 et 26 énoncés après la proposition.

**Proposition 24.** *Soit  $\Omega \subseteq \mathbb{R}^{d+1}$  et posons  $\Gamma_\varepsilon := \Omega \cap (\varepsilon \mathbb{Z}^d \times \varepsilon^2 \mathbb{Z})$ . Supposons que  $u \in C^{0,\alpha}(\overline{\Omega})$  et  $v : \Gamma_\varepsilon \rightarrow \mathbb{R}$  satisfont respectivement*

$$\partial_t u + \mathcal{P}_{1,\Lambda}^-(D^2 u) \leq 1 \quad \text{in } \Omega$$

et

$$\hat{\partial}_t^\varepsilon v + F_\varepsilon[v] \geq -1 \quad \text{in } \Gamma_\varepsilon.$$

Soit  $0 < \delta \leq 1$ . On définit, pour tout  $(p, q, h, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ , une fonction  $\Phi : \Omega \times \Gamma_\varepsilon \rightarrow \mathbb{R}$  par

$$\Phi_{p,q,h,k}(x, t, y, s) := u(x, t) - v(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s - h)^2) - p \cdot x - q \cdot y - k.$$

Supposons que  $V \subseteq \mathbb{R}^{d+1} \times (\varepsilon \mathbb{Z}^d \times \varepsilon^2 \mathbb{Z})$  et  $W \subseteq \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$  sont telles que  $\overline{V} \subseteq \Omega \times \Gamma_\varepsilon$  et, pour tout  $(p, q, h, k) \in W$ , il existe  $\hat{k}$  et  $(x, t, y, s) \in V$  tel que

$$0 = \Phi_{p,q,h,\hat{k}}(x, t, y, s) = \sup \{ \Phi_{p,q,h,\hat{k}}(\xi, \tau, \eta, \sigma) : (\xi, \tau, \eta, \sigma) \in \Omega \times \Gamma_\varepsilon, \tau \leq t, s \in [0, T] \}$$

et

$$|\hat{k} - k| \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \delta^{-1/2} \varepsilon^2.$$

Alors il existe une constante  $C > 1$  dépendant seulement de  $d, \Lambda, \|u\|_{C^{0,\alpha}}$  et  $\text{diam } \Omega$  telle que pour tout  $\rho > 0$ ,

$$|W| \leq C \delta^{-2(d+1)} \left( 1 + \frac{\varepsilon^\alpha + \delta^{1/2}}{\rho} \right)^{2d+2} |V + Q_\rho^f \times Q_\rho^f|,$$

où  $Q_\rho^f := B_\rho \times (0, \rho^2)$  est le cylindre parabolique forward en temps centré en  $(0, 0)$ .

Nous pouvons formuler quelques commentaires sur cette proposition. Il est maintenant nécessaire d'introduire un espace de paramètres de dimension  $2d + 2$  (au lieu de  $2d$  dans le cas elliptique) :

- Les paramètres  $p$  et  $q$  permettent de faire varier les pentes spatiales des enveloppes monotones (qui sont convexes en espace) ;
- Le paramètre  $h$  permet de translater en temps les fonctions  $u$  et  $v$  l'une par rapport à l'autre ;
- le paramètre  $k$  permet d'ajuster la hauteur pour laquelle le supremum est atteint.

Nous pouvons faire varier de façon continue les paramètres  $p, q$  et  $h$  dans des intervalles mais nous devons faire attention à la variable  $k$  qui sert à ajuster la hauteur du sup. En effet, comme la fonction  $v$  est à valeurs discrète la fonction  $\Phi_{p,q,h,k}$  est à valeurs discrètes. Il en découle que le niveau  $\hat{k}$  qui permet d'ajuster le sup au point de contact  $(x, t, y, s)$  à la valeur 0 prend ses valeurs possibles dans un ensemble discret (dépendant de  $p, q$  et  $h$ ) et ne donc pas être choisi de façon continue. De plus, cette estimation

repose sur l'estimation suivante. Etant donné des points réalisant le sup, nous avons l'inégalité suivante : pour une certaine constante  $C > 0$ ,

$$|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2 \leq C\delta^{-2}(\varepsilon^{2\alpha} + \delta + r^2).$$

avec  $r$  donné par

$$r := (|x_1 - x_2|^2 + |y_1 - y_2|^2 + |t_1 - t_2| + |s_1 - s_2|)^{1/2}.$$

En utilisant ensuite un lemme de mesure, nous en déduisons alors l'estimation de la proposition.

La preuve du taux de convergence s'appuie de manière cruciale sur une forme parabolique de l'inégalité ABP.

**Lemme 25.** *Supposons que  $\Omega \subseteq \mathbb{R}^{d+1}$ ,  $R > 0$ ,  $Q_{4R} \subseteq \Omega$  et  $u \in C(\Omega)$  satisfait*

$$\partial_t u + \mathcal{P}_{1,\Lambda}^+(D^2 u) \geq -1 \quad \text{in } \Omega.$$

*Supposons que  $(x_0, t_0), (x_1, t_1) \in Q_R \cap \{u = \Gamma_u\}$  et  $(p_i, h_i) \in \partial\Gamma_u(x_i, t_i)$  for  $i = 0, 1$ . Alors il existe une constante  $C(d, \Lambda) > 1$  telle que*

$$|p_1 - p_0|^2 + |h_1 - h_0|^2 \leq C(|x_1 - x_0|^2 + |t_1 - t_0|^2 + |x_1 - x_0|^4).$$

**Lemme 26.** *Supposons que  $\mathcal{F}$  satisfait (F1)-(F3) et  $F_\varepsilon$  satisfait (S2),  $\Lambda \subseteq \mathbb{R}^{d+1}$ ,  $R > 1$ ,  $Q_{4R} \cap \mathbb{Z}^{d+1} \subseteq \Lambda$  et  $u : \Lambda \rightarrow \mathbb{R}$  satisfait*

$$\hat{\partial}_t^\varepsilon u + F_\varepsilon[u] \geq -1 \quad \text{sur } \Lambda.$$

*Supposons que, pour chaque  $i \in \{0, 1\}$ ,  $(z_i, t_i) \in Q_R \cap \Lambda \cap \{u = \Gamma_u\}$  et  $(p_i, h_i) \in \partial\Gamma_u(z_i, t_i)$ . Alors il existe  $C(d, \Lambda, K_0, K_1) > 1$  tel que si  $R \geq C\varepsilon$  alors*

$$|p_1 - p_0|^2 + |h_1 - h_0|^2 \leq C(\varepsilon^2 + |z_1 - z_0|^2 + |t_1 - t_0|^2 + |z_1 - z_0|^4).$$

La stratégie de la preuve du taux de convergence est la suivante :

**Etape 1 :** On s'intéresse à la famille de fonctions  $\Phi_{p,q,h,k} : \Omega \times \Gamma_\varepsilon$  donnée par :

$$\Phi_{p,q,h,k}(x, t, y, s) := u(x, t) - v(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s - h)^2) - p \cdot x - q \cdot y - k.$$

et on considère l'ensemble des points réalisant le sup de manière monotone par rapport au temps  $t$  et global par rapport à la variable discrète  $s$  sur la grille.

**Etape 2 :** On compare la taille de l'ensemble des points réalisant les sup ci-dessus avec la taille de l'ensemble des points admettant un développement quadratique avec un terme cubique contrôlé par l'estimation  $W^{3,\sigma}$  parabolique donnée par le théorème 14.

**Etape 3 :** Si l'erreur  $E$  commise est trop grande, nous pouvons choisir les variables  $p, q, h$  and  $k$  dans des intervalles de longueurs suffisamment grands de sorte qu'il nous est possible – par un argument de mesure utilisant l'estimation  $W^{3,\varepsilon}$  parabolique – de considérer un point  $(x_1, t_1, y_1, s_1) \in \tilde{V}_\rho$  qui a dans un voisinage proche un point  $(\tilde{x}_0, \tilde{t}_0)$ , qui admet la propriété de posséder un développement polynomial cubique global avec un terme cubique contrôlé. En insérant le développement de Taylor du second ordre avec un contrôle sur le terme cubique, nous aboutissons à une contradiction en combinant :

- l'estimation de la dérivation temporelle et de l'opérateur spatial par différences finies ;
- l'uniforme ellipticité de l'opérateur  $\mathcal{F}$  qui contrôle, par une estimation Lipschitz, la perturbation liée au terme cubique du développement de Taylor.

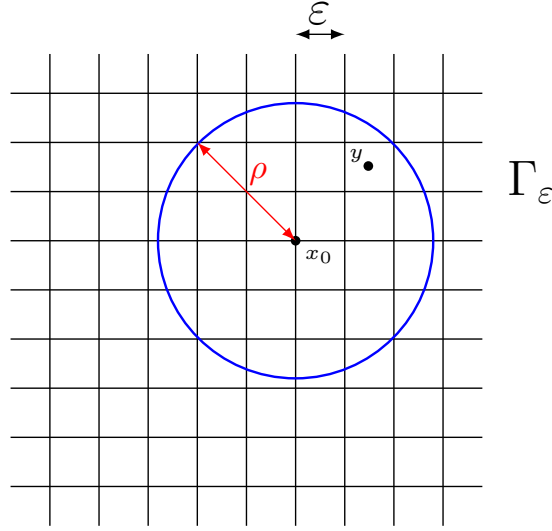


Figure 6 – Preuve du théorème de convergence elliptique (resp. parabolique). Nous trouvons un point  $x_0$  réalisant le sup par l'inégalité ABP et un argument de mesure puis on trouve un point  $y$  situé suffisamment proche tel que  $y$  vérifie l'estimation  $W^{3,\varepsilon}$  elliptique (resp. parabolique).

### 3 Perspectives

Nous allons ici donner quelques pistes de recherche futures en lien avec les résultats obtenus dans la présente thèse.

◊ *Etude des schémas paraboliques.* Pour compléter l'étude faite dans cette thèse, il reste principalement à traiter la question d'obtenir un taux algébrique pour d'autres types de schémas. Nous pensons surtout aux schémas d'Euler explicites en temps, y compris en incorporant d'autres dépendances pour la non-linéarité. La résolution des schémas explicites permettrait d'obtenir un taux de convergence dans le cas du jeu de Mark et Helen dans sa version parabolique. En effet, ce schéma est explicite si  $\mathcal{F}$  dépend de  $u$  et semi-explicite dans le cas contraire. Nous pouvons aussi prendre en compte d'autres dépendances pour la non-linéarité, ce qui est intéressant pour la classe d'équations elliptiques ou paraboliques approchées par les schémas. Ces deux points font actuellement l'objet d'un travail en collaboration avec Scott Armstrong.

◊ *Etude du taux de convergence pour l'asymptotique en temps long pour des équations paraboliques homogènes complètement non-linéaires.* On s'intéresse à l'équation

$$\begin{cases} \partial_t u + \mathcal{F}(D^2 u) = 0 & \text{sur } \mathbb{R}^d \times \mathbb{R}_+, \\ u(x, 0) = g(x), \end{cases}$$

pour un opérateur uniformément elliptique  $\mathcal{F}$  qui est de plus *positivement homogène*. Pour  $\sigma > 0$ , la fonction  $u_\sigma$  donnée par  $u_\sigma(x, t) := \sigma^\alpha u(\sigma^{1/2}x, \sigma t)$  est une solution de (0.3). Dans [7], Armstrong et Trokhimtchouk ont obtenu la convergence, localement uniforme, de la limite rééchelonnée de la solution du problème de Cauchy pour des données initiales positives (ou négatives) vers une gaussienne  $\Phi^+$  ( $\Phi^-$ ) sur  $\mathbb{R}^d \times \mathbb{R}_+$ . Il serait intéressant de quantifier ce taux de convergence.

◊ *Homogénéisation stochastique pour des équations paraboliques.* Tout d'abord, on peut prouver des résultats de régularité et d'homogénéisation stochastique pour des équations paraboliques complètement non-linéaires qui ne sont pas sous forme divergente et *sans* l'hypothèse d'uniforme ellipticité en utilisant certains outils déjà développés dans les chapitres 2 et 4. Typiquement, les équations que nous avons en tête sont strictement elliptiques mais peuvent présenter des constantes d'ellipticité qui sont arbitrairement grandes et oscillantes à l'échelle microscopique.

On peut analyser des équations paraboliques non-linéaires de la forme

$$\partial_t u^\varepsilon + \mathcal{F}\left(\omega, \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, D^2 u^\varepsilon\right) = 0, \quad (78)$$

où la non-linéarité  $\mathcal{F}$  présente une *ellipticité localement uniforme* : il existe une constante  $\Lambda \geq 1$  et une variable aléatoire  $\lambda$  telle que

$$\mathcal{P}_{\lambda(\omega), \Lambda}^-(M - N) \leq \mathcal{F}(\omega, s, y, M) - \mathcal{F}(\omega, s, y, N) \leq \mathcal{P}_{\lambda(\omega), \Lambda}^+(M - N).$$

De plus, la variable aléatoire  $\lambda$  satisfait une certaine condition de borne sur l'un de ses moments.

Par ailleurs, sous l'hypothèse d'*uniforme ellipticité*, une autre question serait de quantifier le taux de convergence vers l'équation homogénéisée. Il est connu que, sous des conditions au bord appropriées, les solutions  $u^\varepsilon$  convergent localement uniformément et avec probabilité 1, lorsque  $\varepsilon \rightarrow 0$ , vers la solution  $u$  de l'équation déterministe

$$\partial_t u + \bar{\mathcal{F}}(t, x, D^2 u) = 0. \quad (79)$$

D'un point de vue théorique ou pratique, il est intéressant de quantifier à quel point  $\varepsilon$  doit être choisi petit de sorte à garantir que (78) est une bonne approximation de (79). En termes de vitesse de convergence vers la limite homogénéisée, on s'attend à l'estimée suivante

$$\mathbb{P}\left[\sup_{x \in \Omega} |u^\varepsilon(x, t) - u(x, t)| \geq C\varepsilon^\alpha\right] \leq C \exp(-\varepsilon^{-1}).$$

Le cas elliptique pour ces deux résultats a été récemment étudiée par Armstrong et Smart dans [5] et [4].

◊ *Interprétation des EDPs par des jeux.* Un panorama non exhaustif des liens entre EDP et jeux a été donné à la table 1. Une interprétation par jeux fournit une représentation asymptotique de la solution d'une EDP. Cette approximation de la solution de l'EDP au sens de viscosité peut parfois être quantifié (voir le chapitre 3 dans le cas elliptique du jeu de Mark et Helen). De plus, les propriétés d'une équation peuvent aussi être étudiées à travers le jeu. Dans le cas du mouvement par courbure moyenne, Liu [110] a étudié la propriété d'épaississement ("fattening") pour le mouvement par courbure moyenne en interprétant de manière géométrique la dynamique d'une particule dans le jeu avec les ensembles de niveau de la solution de l'EDP. On s'attend que l'interprétation par jeu peut aussi être intéressante pour étudier certains phénomènes dans les problèmes portant sur la propagation de fronts comme les états asymptotiques des équations de réaction-diffusion semilinéaires (voir [21]).

D'autres directions de recherche reposent sur les jeux eux-mêmes. On peut étudier la convergence des jeux lorsqu'un paramètre change la nature des EDP. Dans [77], Serfaty et Imbert ont construit des jeux associés au flot de la courbure intégrale de la mesure singulière mais les règles obtenues sont beaucoup plus compliquées que celles du jeu de Paul et Carol pour le mouvement à courbure moyenne (données à la section 0.1.3.1). Il est connu [76] que le flot de la courbure intégrale converge le flot de la courbure moyenne lorsque  $\alpha \rightarrow 1$  et c'est un problème naturel de trouver des jeux plus simples qui convergent dans un certain sens qu'il reste à préciser vers le jeu de Paul et Carol. La même question peut être posée pour les opérateurs de type Laplacien fractionnaires dans le cas des équations intégral-différentielles partielles. Plus largement, la question sous-jacente, qui dépasse l'interprétation par jeux, est :

- Étudier la représentation d'une famille de problèmes  $(P_\varepsilon)$  dépendant d'un paramètre  $\sigma$  lorsqu'on fait tendre  $\sigma$  vers 0. Cela forme un problème de double passage à la limite lorsqu'on fait tendre le paramètre  $\varepsilon$  vers 0 puis  $\sigma$  vers 0 ou bien  $\sigma$  vers 0 puis  $\varepsilon$  vers 0. Un résultat dans ce sens a été donné par Barles, Bronsard et Souganidis [9].

Les jeux considérés dans la table 1 ont fait intervenir jusqu'à maintenant seulement deux joueurs mais on pourrait considérer le cas d'un nombre fini  $N \geq 2$  ou un nombre infini de joueurs. Les jeux à champ moyen proposés par Lasry et Lions (voir [103] et aussi [101, 102]) dans le cadre stochastique constituent un premier type d'approche possible. Le comportement commun des joueurs au sein de la communauté est décrit par une condition d'équilibre de Nash. Il semble naturel d'étudier les jeux présentés à la table 1



dans ce cadre. Pour les jeux à champ moyen, nous renvoyons au papier récapitulatif [103] et à [101, 102] ainsi qu'aux deux articles de Cardaliaguet [38, 39]. Il serait également intéressant de chercher des versions déterministes des jeux à champ moyen.

# A game interpretation of the Neumann problem for fully nonlinear parabolic and elliptic equations

Ce travail fournit une interprétation par contrôle déterministe pour une classe étendue d'équations non-linéaires elliptiques et paraboliques avec des conditions de Neumann au bord sur un domaine régulier. Nous construisons des familles de jeux répétés à deux personnes dépendant d'un petit paramètre  $\varepsilon$  qui étend ceux proposés par Kohn et Serfaty [88]. Ces nouveaux jeux résolvent une condition de Neumann au bord en introduisant des règles spécifiques au voisinage du bord. Nous montrons que la fonction valeur converge, au sens de viscosité, vers la solution de l'EDP lorsque  $\varepsilon$  tend vers zéro. De plus, notre construction permet également d'interpréter d'une part des conditions de type oblique et d'autre part des conditions mixtes Dirichlet-Neumann au bord.

Jean-Paul Daniel. A game interpretation of the Neumann problem for fully nonlinear parabolic and elliptic equations. Article publié dans <i>ESAIM: Control, Optimisation and Calculus of Variations</i> , Vol. 19, no 4 (2013), pp 1109-1165. doi:10.1051/cocv/2013047.
---



# Chapter 1

## A game interpretation of the Neumann problem for fully nonlinear parabolic and elliptic equations

### Abstract

We provide a deterministic-control-based interpretation for a broad class of fully nonlinear parabolic and elliptic PDEs with continuous Neumann boundary conditions in a smooth domain. We construct families of two-person games depending on a small parameter  $\varepsilon$  which extend those proposed by Kohn and Serfaty [88]. These new games treat a Neumann boundary condition by introducing some specific rules near the boundary. We show that the value function converges, in the viscosity sense, to the solution of the PDE as  $\varepsilon$  tends to zero. Moreover, our construction allows us to treat both the oblique and the mixed type Dirichlet-Neumann boundary conditions.

### Contents

---

<b>1.1</b>	<b>Introduction</b>	<b>48</b>
<b>1.2</b>	<b>The games</b>	<b>52</b>
1.2.1	The linear heat equation	52
1.2.2	General parabolic equations	55
1.2.3	Nonlinear elliptic equations	66
<b>1.3</b>	<b>Convergence</b>	<b>70</b>
1.3.1	Viscosity solutions with Neumann condition	70
1.3.2	The parabolic case	72
1.3.3	The elliptic case	77
<b>1.4</b>	<b>Consistency</b>	<b>78</b>
1.4.1	The parabolic case	79
1.4.2	Proof of Lemma 1.4.9 and Proposition 1.4.10	86
1.4.3	Application to stability	94
1.4.4	The elliptic case	96
<b>1.5</b>	<b>Stability</b>	<b>99</b>
1.5.1	The parabolic case	99
1.5.2	The elliptic case	100
<b>1.6</b>	<b>Some natural generalizations</b>	<b>103</b>
1.6.1	Elliptic PDE with mixed Dirichlet-Neumann boundary conditions	103
1.6.2	Parabolic PDE with an oblique boundary condition	104

---

**Keywords:** Fully nonlinear elliptic equations, viscosity solutions, Neumann problem, deterministic control, optimal control, dynamic programming principle, oblique problem, mixed-type Dirichlet-Neumann boundary conditions.

**Mathematics Subject Classification** 49L25, 35J60, 35K55, 49L20, 35D40, 35M12, 49N90.

## 1.1 Introduction

In this paper, we propose a deterministic control interpretation, via “two persons repeated games”, for a broad class of fully nonlinear equations of elliptic or parabolic type with a continuous Neumann boundary condition in a smooth (not necessarily bounded) domain. In their seminal paper [88], Kohn and Serfaty focused on the one hand on the whole space case in the parabolic setting and on the other hand on the Dirichlet problem in the elliptic framework. They construct a monotone and consistent difference approximation of the operator from the dynamic programming principle associated to the game.

Our motivation here is to adapt their approach to the Neumann problem in both settings. Furthermore, once this issue is solved, we will see how the oblique or the mixed type Dirichlet-Neumann boundary problem can also be treated by this analysis. We consider equations in a domain  $\Omega \subset \mathbb{R}^d$  having the form

$$-u_t + \mathcal{F}(t, x, u, Du, D^2u) = 0 \quad (1.1.1)$$

or

$$\mathcal{F}(x, u, Du, D^2u) + \lambda u = 0, \quad (1.1.2)$$

where  $\mathcal{F}$  is elliptic in the sense that  $\mathcal{F}$  is monotone in its last variable, subject to the Neumann boundary condition

$$\frac{\partial u}{\partial n} = h. \quad (1.1.3)$$

As in [88], the class of functions  $\mathcal{F}$  considered is large, including those that are non-monotone in the  $u$  argument and degenerate in the  $D^2u$  argument. We make the same hypotheses on the continuity, growth, and  $u$ -dependence of  $\mathcal{F}$  imposed in [88]. They are recalled at the end of the section. In the stationary setting (1.1.2), we focus on the Neumann problem, solving the equation in a domain  $\Omega$  with (1.1.3) at  $\partial\Omega$ . In the time-dependent setting (1.1.1), we address the Cauchy problem, solving the equation with (1.1.3) at  $\partial\Omega$  for  $t < T$  and  $u = g$  at terminal time  $t = T$ . The PDEs and boundary conditions are always interpreted in the “viscosity sense” (Section 1.3 presents a review of this notion).

Our games have two opposite players, Helen and Mark, who always make decisions rationally and deterministically. The rules depend on the form of the equation, but there is always a small parameter  $\varepsilon$ , which governs the spatial step size and (in time-dependent problems) the time step. Helen’s goal is to optimize her worst-case outcome. When  $\mathcal{F}$  is independent of  $u$ , we shall characterize her value function  $u^\varepsilon$  by the dynamic programming principle. If  $\mathcal{F}$  depends also on  $u$ , the technicality of our arguments requires to introduce a level-set formulation since the uniqueness of the viscosity solution is no longer guaranteed. The score  $U^\varepsilon$  of Helen now depends on a new parameter  $z \in \mathbb{R}$ . In the parabolic setting, it is defined by an induction backward in time given by

$$\forall z \in \mathbb{R}, \quad U^\varepsilon(x, z, t) = \max_{p, \Gamma} \min_{\Delta \hat{x}} U^\varepsilon(x + \Delta x, z + \Delta z, t + \Delta t),$$

endowed with the final-time condition  $U^\varepsilon(x, z, t) = g(x) - z$ . The max on  $p, \Gamma$ , the min on  $\Delta \hat{x}$  as well as the definitions of  $\Delta x, \Delta z$  and  $\Delta t$  are given by some constraints depending on the rules of the game and some powers of  $\varepsilon$ . This dynamic programming principle is similar to the one given in [88, Section 2.3]. In that case, our value functions  $u^\varepsilon$  of interest are defined through the 0-level set of  $U^\varepsilon$  with respect to  $z$  as the maximal and the minimal solutions of  $U^\varepsilon(x, z, t) = 0$ . They satisfy two dynamic programming

inequalities (for the details of our games and the definition of Helen's value function, see Section 1.2). Roughly speaking, our main result states that

$$\limsup_{\varepsilon \rightarrow 0} u^\varepsilon \text{ is a viscosity subsolution of the PDE, and}$$

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon \text{ is a viscosity supersolution of the PDE.}$$

For the general theory of viscosity solutions to fully nonlinear equations with Neumann (or oblique) boundary condition the reader is referred to [45, 10, 79]. As for the Neumann boundary condition, its relaxation in the viscosity sense was first proposed by Lions [106].

Our result is most interesting when the PDE has a comparison principle, i.e. when every subsolution must lie below any supersolution. For such equations, we conclude that  $\lim u^\varepsilon$  exists and is the unique viscosity solution of the PDE. In the case when  $\mathcal{F}$  is continuous in all its variable, there are already a lot of comparison and existence results for viscosity solutions of second order parabolic PDEs with general Neumann type boundary conditions. We refer for this to [10, 12, 106, 79] and references therein. For homogeneous Neumann conditions, Sato [125] has obtained such a comparison principle for certain parabolic PDEs.

We are interested here in giving a game interpretation for fully nonlinear parabolic and elliptic equations with a Neumann condition. Applications of the Neumann condition to deterministic optimal control and differential games theory in [106] rely much on a reflection process, the solution of the deterministic Skorokhod problem. Its properties in different situations are studied in many articles such as [138, 108, 57]. The case of the Neumann problem for the motion by mean curvature was studied by Giga and Liu [73]. There, a billiard game was introduced to extend the interpretation made by Kohn and Serfaty [87] via the game of Paul and Carol. It was based on the natural idea that a homogeneous Neumann condition will be well-modeled by a reflection on the boundary. Liu also applies this billiard dynamics to study some first order Hamilton-Jacobi equations with Neumann or oblique boundary conditions [109]. Nevertheless, in our case, if we want to give a billiard interpretation with a bouncing rule which can send the particle far from the boundary, we can only manage to solve the homogeneous case. This is not too surprising because the reflection across  $\partial\Omega$  is precisely associated to a homogeneous Neumann condition.

Another approach linked to the Neumann condition is to proceed by penalization on the dynamics. For a bounded convex domain, Lions, Menaldi and Sznitman [107] construct a sequence of stochastic differential equations with a term in the drift coefficients that strongly penalizes the process from leaving the domain. Its solution converges towards a diffusion process which reflects across the boundary with respect to the normal vector. Barles and Lions [17] also treat the oblique case by precisely establishing the links between some approximated processes and the elliptic operators associated to the original oblique stochastic dynamics.

Instead of a billiard, our approach here proceeds by a suitable penalization on the dynamics depending on the Neumann boundary condition. It will be favorable to one player or the other according to its sign. We modify the rules of the game only in a small neighborhood of the boundary. The particle driven by the players can leave the domain but then it is projected within. This particular move, combined with a proper weight associated to the Neumann boundary condition, gives the required penalization. Outside this region, the usual rules are conserved. Therefore the previous analysis within  $\Omega$  done by Kohn and Serfaty can be preserved. We focus all along this article on the changes near the boundary and their consequences on the global convergence theorem. In this context, the modification of the rules of the original game introduces many additional difficulties intervening at the different steps of the proof. Most of all, they are due to the geometry of the domain or the distance to the boundary. As a result, our games seem like a natural adaptation of the games proposed by Kohn and Serfaty by permitting to solve an inhomogeneous Neumann condition  $h$  depending on  $x$  on the boundary. We only require  $h$  to be continuous and uniformly bounded, the domain to be  $C^2$  and to satisfy some natural geometric conditions in order to ensure the well-posedness of our games. Moreover our approach can easily be extended both to the oblique and the mixed Neumann-Dirichlet boundary conditions in both parabolic and elliptic

settings. Our games can be compared to those proposed in [88] for the elliptic Dirichlet problem: if the particle crosses the boundary, the game is immediately stopped and Helen receives a bonus  $b(x_F)$  where  $b$  corresponds to the Dirichlet boundary condition and  $x_F$  is the final position. Meanwhile, our games cannot stop unexpectedly, no matter the boundary is crossed or not.

Our games, like the ones proposed by Kohn and Serfaty, are deterministic but closely related to a recently developed stochastic representation due to Cheridito, Soner, Touzi and Victoir [42] (their work uses a backward stochastic differential equation, BSDE, whose structure depends on the form of the equation).

Another interpretation is to look our games as a numerical scheme whose solution is an approximation of a solution of a certain PDE. This aspect is classical and has already been exploited in several contexts. We mention the work of Peres, Schramm, Sheffield and Wilson [121] who showed that the infinity Laplace equation describes the continuum limit of the value function of a two-player, random-turn game called  $\varepsilon$ -step tug-of-war. In related work, Armstrong, Smart and Sommersille [6] obtained existence, uniqueness and stability results for an infinity Laplace equation with mixed Dirichlet-Neumann boundary terms by comparing solutions of the PDE to subsolutions and supersolutions of a certain finite difference scheme, by following a previous work of Armstrong and Smart for the Dirichlet case [3].

This paper is organized as follows:

- Section 1.2 presents the two-person games that we associate with the PDEs (1.1.1) and (1.1.2), motivating and stating our main results. The section starts with a simple case before addressing the general one. Understanding our games is still easy, though the technicality of our proofs is increased. Since  $\mathcal{F}$  depends on  $u$ , the game determines a pair of value functions  $u^\varepsilon$  and  $v^\varepsilon$ . Section 1.2.2.1 gives a formal argument linking the principle of dynamic programming to the PDE in the limit  $\varepsilon \rightarrow 0$  and giving the optimal strategies for Helen that will be essential to obtain consistency at Section 1.4.
- Section 1.3 addresses the link between our game and the PDE with full rigor. The proofs of convergence follow the background method of Barles and Souganidis [20], i.e. they use the stability, monotonicity and consistency of the schemes provided by our games. Their theorem states that if a numerical scheme is monotone, stable, and consistent, then the associated “lower semi-relaxed limit” is a viscosity supersolution and the associated “upper semi-relaxed limit” is a viscosity subsolution. The main result in Section 1.3 is a specialization of their theorem in our framework: if  $v^\varepsilon$  and  $u^\varepsilon$  remain bounded as  $\varepsilon \rightarrow 0$  then the lower relaxed semi-limit of  $v^\varepsilon$  is a viscosity supersolution and the upper relaxed semi-limit of  $u^\varepsilon$  is a viscosity subsolution. We also have  $v^\varepsilon \leq u^\varepsilon$  with no extra hypothesis in the parabolic setting, or if  $\mathcal{F}$  is monotone in  $u$  in the elliptic setting. If the PDE has a comparison principle (see [20]) then it follows that  $\lim u^\varepsilon = \lim v^\varepsilon$  exists and is the unique viscosity solution of the PDE.
- The analysis in Section 1.3 shows that consistency and stability imply convergence. Sections 1.4 and 1.5 provide the required consistency and stability results. The new difficulties due to the penalization corresponding to the Neumann condition arise here. The main difficulty is to control the degeneration of the consistency estimate obtained in [88] with respect to the penalization. Therefore we will mainly focus on the consistency estimates whereas the needed changes for stability will be simply indicated.
- Section 1.6 describes the games associated on the one hand to the oblique problem in the parabolic setting and on the other hand to the mixed type Dirichlet-Neumann boundary conditions in the elliptic framework. By combining the results associated to the game associated to the Neumann problem in Section 1.2 with the ideas already presented in [88], we can obtain the results of convergence.

**Notation:** The term domain will be reserved for a nonempty, connected, and open subset of  $\mathbb{R}^d$ . If  $x, y \in \mathbb{R}^d$ ,  $\langle x, y \rangle$  denotes the usual Euclidean inner product and  $\|x\|$  the Euclidean length of  $x$ . If  $A$  is a  $d \times d$  matrix,  $\|A\|$  denotes the operator norm  $\|A\| := \sup\{\|Ax\| : \|x\| \leq 1\}$ .  $\mathcal{S}_d$  denotes the set of symmetric  $d \times d$  matrices and  $E_{ij}$  the  $(i, j)$ -th matrix unit, the matrix whose only nonzero element is equal to 1 and occupies the  $(i, j)$ -th position.

Let  $\mathcal{O}$  be a domain in  $\mathbb{R}^d$  and  $C_b^k(\mathcal{O})$  be the vector space of  $k$ -times continuously differentiable functions  $u: \mathcal{O} \rightarrow \mathbb{R}$ , such that all the partial derivatives of  $u$  up to order  $k$  are bounded on  $\mathcal{O}$ . For a domain  $\Omega$ , we define

$$C_b^k(\overline{\Omega}) := \left\{ u \in L^\infty(\overline{\Omega}) : \exists \mathcal{O} \supset \overline{\Omega}, \mathcal{O} \text{ domain}, \exists v \in C_b^k(\mathcal{O}) \text{ s.t. } u = v|_{\overline{\Omega}} \right\}.$$

It is equipped with the norm  $\|\cdot\|_{C_b^k(\overline{\Omega})}$  given by  $\|\phi\|_{C_b^k(\overline{\Omega})} := \sum_{i=0}^k \|D^i \phi\|_{L^\infty(\overline{\Omega})}$ .

If  $\Omega$  is a smooth domain, say  $C^2$ , the distance function to  $\partial\Omega$  is denoted by  $\text{dist} = \text{dist}(\cdot, \partial\Omega)$ , and we recall that, for all  $x \in \partial\Omega$ , the outward normal  $n(x)$  to  $\partial\Omega$  at  $x$  is given by  $n(x) = -D \text{dist}(x)$ .

Observe that, if  $\partial\Omega$  is assumed to be bounded and at least of class  $C^2$ , any  $x \in \mathbb{R}^d$  lying in a sufficiently small neighborhood of the boundary admits a unique projection onto  $\partial\Omega$ , denoted by

$$\bar{x} := \text{proj}_{\partial\Omega}(x).$$

In particular, the vector  $x - \bar{x}$  is parallel to  $n(\bar{x})$ . The projection onto  $\overline{\Omega}$  will be denoted by  $\text{proj}_{\overline{\Omega}}$ . When it is well-defined, it can be decomposed as

$$\text{proj}_{\overline{\Omega}}(x) = \begin{cases} \text{proj}_{\partial\Omega}(x), & \text{if } x \notin \Omega, \\ x, & \text{if } x \in \Omega. \end{cases}$$

For each  $a > 0$ , we define  $\Omega(a) = \{x \in \overline{\Omega}, \text{dist}(x) < a\}$ . We recall the following classical geometric condition (see e.g. [61]).

**Definition 1.1.1** (Interior ball condition). *The domain  $\Omega$  satisfies the interior ball condition at  $x_0 \in \partial\Omega$  if there exists an open ball  $B \subset \Omega$  with  $x_0 \in \partial B$ .*

We close this introduction by listing our main hypotheses on the form of the PDE. First of all we precise some hypotheses on the domain  $\Omega$ . Throughout this article,  $\Omega$  will denote a  $C^2$ -domain. In the unbounded case, we impose the following slightly stronger condition than the interior ball condition.

**Definition 1.1.2** (Uniform interior/exterior ball condition). *The domain  $\Omega$  satisfies the uniform interior ball condition if there exists  $r > 0$  such that for all  $x \in \partial\Omega$  there exists an open ball  $B \subset \Omega$  with  $x \in \partial B$  and radius  $r$ . Moreover, the domain  $\Omega$  satisfies the uniform exterior ball condition if  $\mathbb{R}^d \setminus \overline{\Omega}$  satisfies the uniform interior ball condition.*

We observe that the uniform interior ball condition implies the interior ball condition and that both the uniform interior and exterior ball conditions hold automatically for a  $C^2$ -bounded domain.

The Neumann boundary condition  $h$  is assumed to be continuous and uniformly bounded on  $\partial\Omega$ . Similarly, in the parabolic framework, the final-time data  $g$  is supposed to be continuous and uniformly bounded on  $\overline{\Omega}$ .

The real-valued function  $\mathcal{F}$  in (1.1.1) is defined on  $\mathbb{R} \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$ . It is assumed throughout to be a continuous function of all its variables, and also that

- $\mathcal{F}$  is monotone in  $\Gamma$  in the sense that

$$\mathcal{F}(t, x, z, p, \Gamma_1 + \Gamma_2) \leq \mathcal{F}(t, x, z, p, \Gamma_1) \quad \text{for } \Gamma_2 \geq 0. \quad (1.1.4)$$

In the time-dependent setting (1.1.1) we permit  $\mathcal{F}$  to grow linearly in  $|z|$  (so solutions can grow exponentially, but cannot blow up). However we require uniform control in  $x$  (so solutions remain bounded as  $\|x\| \rightarrow \infty$  with  $t$  fixed). In fact we assume that

- $\mathcal{F}$  has at most linear growth in  $z$  near  $p = 0$ ,  $\Gamma = 0$ , in the sense that for any  $K$  we have

$$|\mathcal{F}(t, x, z, p, \Gamma)| \leq C_K(1 + |z|), \quad (1.1.5)$$

for some constant  $C_K \geq 0$ , for all  $x \in \overline{\Omega}$  and  $t, z \in \mathbb{R}$ , when  $\|(p, \Gamma)\| \leq K$ .



- $\mathcal{F}$  is locally Lipschitz in  $p$  and  $\Gamma$  in the sense that for any  $K$  we have

$$|\mathcal{F}(t, x, z, p, \Gamma) - \mathcal{F}(t, x, z, p', \Gamma')| \leq C_K(1 + |z|)\|(p, \Gamma) - (p', \Gamma')\|, \quad (1.1.6)$$

for some constant  $C_K \geq 0$ , for all  $x \in \overline{\Omega}$  and  $t, z \in \mathbb{R}$ , when  $\|(p, \Gamma)\| + \|(p', \Gamma')\| \leq K$ .

- $\mathcal{F}$  has controlled growth with respect to  $p$  and  $\Gamma$ , in the sense that for some constants  $q, r \geq 1$ ,  $C > 0$ , we have

$$|\mathcal{F}(t, x, z, p, \Gamma)| \leq C(1 + |z| + \|p\|^q + \|\Gamma\|^r), \quad (1.1.7)$$

for all  $t, x, z, p$  and  $\Gamma$ .

In the stationary setting (1.1.2) our solutions will be uniformly bounded. To prove the existence of such solutions we need the discounting to be sufficiently large. We also need analogues of (1.1.6)–(1.1.7) but they can be local in  $z$  since  $z$  will ultimately be restricted to a compact set. In fact, we assume that

- There exists  $\eta > 0$  such that for all  $K \geq 0$ , there exists  $C_K^* > 0$  satisfying

$$|\mathcal{F}(x, z, p, \Gamma)| \leq (\lambda - \eta)|z| + C_K^*, \quad (1.1.8)$$

for all  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$ , when  $\|(p, \Gamma)\| \leq K$ ; here  $\lambda$  is the coefficient of  $u$  in the equation (1.1.2).

- $\mathcal{F}$  is locally Lipschitz in  $p$  and  $\Gamma$  in the sense that for any  $K$  and  $L$  we have

$$|\mathcal{F}(x, z, p, \Gamma) - \mathcal{F}(x, z, p', \Gamma')| \leq C_{K,L}\|(p, \Gamma) - (p', \Gamma')\|, \quad (1.1.9)$$

for some constant  $C_{K,L} \geq 0$ , for all  $x \in \overline{\Omega}$ , when  $\|(p, \Gamma)\| + \|(p', \Gamma')\| \leq K$  and  $|z| \leq L$ .

- $\mathcal{F}$  has controlled growth with respect to  $p$  and  $\Gamma$ , in the sense that for some constants  $q, r \geq 1$  and for any  $L$  we have

$$|\mathcal{F}(x, z, p, \Gamma)| \leq C_L(1 + \|p\|^q + \|\Gamma\|^r), \quad (1.1.10)$$

for some constant  $C_L \geq 0$ , for all  $x, p$  and  $\Gamma$ , and any  $|z| \leq L$ .

## 1.2 The games

This section presents our games. We begin by dealing with the linear heat equation. Section 1.1.1 addresses the time-dependent problem depending non linearly on  $u$ ; our main rigorous result for the time-dependent setting is stated here (Theorem 1.2.4). Section 1.1.2 discusses the stationary setting and states our main rigorous result for that case (Theorem 1.2.7).

### 1.2.1 The linear heat equation

This section offers a deterministic two-persons game approach to the linear heat equation in one space dimension. More precisely, let  $a < c$  and  $\Omega = ]a, c[$ . We consider the linear heat equation on  $\Omega$  with continuous final-time data  $g$  and Neumann boundary condition  $h$  given by

$$\begin{cases} u_t + u_{xx} = 0, & \text{for } x \in \Omega \text{ and } t < T, \\ \frac{\partial u}{\partial n}(x, t) = h(x), & \text{for } x \in \partial\Omega = \{a, c\} \text{ and } t < T, \\ u(x, T) = g(x), & \text{for } x \in \overline{\Omega} \text{ and } t = T. \end{cases} \quad (1.2.1)$$

Our goal is to capture, in the simplest possible setting, how a homogeneous Neumann condition can be retrieved through a repeated deterministic game. The game discussed here shares many features with the ones we will introduce in Sections 1.2.2–1.2.3, though it is not a special case. In particular, it allows to understand the way we need to modify the rules of the pioneering games proposed by Kohn and Serfaty in [88] in order to model the Neumann boundary condition.

There are two players, we call them Mark and Helen. A small parameter  $\varepsilon > 0$  is fixed as are the final time  $T$ , “Helen’s payoff” (a continuous function  $g: [a, c] \rightarrow \mathbb{R}$ ) and a “coupon profile” close to the boundary (a function  $h: \{a, c\} \rightarrow \mathbb{R}$ ). The state of the game is described by its “spatial position”  $x \in \overline{\Omega}$  and “Helen’s score”  $y \in \mathbb{R}$ . We suppose the game begins at time  $t_0$ . Since time steps are increments of  $\varepsilon^2$ , it is convenient to assume that  $T - t_0 = K\varepsilon^2$ , for some  $K$ .

When the game begins, the position can have any value  $x_0 \in \overline{\Omega}$ ; Helen’s initial score is  $y_0 = 0$ . The rules are as follows: if, at time  $t_j = t_0 + j\varepsilon^2$ , the position is  $x_j$  and Helen’s score is  $y_j$ , then

**Step 1:** Helen chooses a real number  $p_j$ .

**Step 2:** After seeing Helen’s choice, Mark chooses  $b_j = \pm 1$  which gives an intermediate position  $\hat{x}_{j+1} = x_j + \Delta\hat{x}_j$  where

$$\Delta\hat{x}_j = \sqrt{2}\varepsilon b_j \in \mathbb{R}.$$

This position  $\hat{x}_{j+1}$  determines the next position  $x_{j+1} = x_j + \Delta x_j$  at time  $t_{j+1}$  by the rule

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}) \in \overline{\Omega},$$

and Helen’s score changes to

$$y_{j+1} = y_j + p_j \Delta\hat{x}_j - \|x_{j+1} - \hat{x}_{j+1}\| h(x_j + \Delta x_j). \quad (1.2.2)$$

**Step 3:** The clock moves forward to  $t_{j+1} = t_j + \varepsilon^2$  and the process repeats, stopping when  $t_K = T$ .

**Step 4:** At the final time  $t_K = T$  a bonus  $g(x_K)$  is added to Helen’s score, where  $x_K$  is the final-time position.

**Remark 1.2.1.** To give a sense to (1.2.2) for all  $\Delta x_j$ , the function  $h$ , which is defined only on  $\{a, c\}$ , can be extended on  $]a, c[$  by any function  $\Omega \rightarrow \mathbb{R}$  since  $\|x_{j+1} - \hat{x}_{j+1}\|$  is different from zero if and only if  $\hat{x}_{j+1} \notin \overline{\Omega}$ . Moreover, by comparing the two moves  $\Delta\hat{x}_j$  and  $\Delta x_j$ , it is clear that  $\|x_{j+1} - \hat{x}_{j+1}\| = \|\Delta x_j - \Delta\hat{x}_j\|$ .

Helen’s goal is to maximize her final score, while Mark’s goal is to obstruct her. We are interested in Helen’s “value function”  $u^\varepsilon(x_0, t_0)$ , defined formally as her maximum worst-case final score starting from  $x_0$  at time  $t_0$ . It is determined by the dynamic programming principle

$$u^\varepsilon(x, t_j) = \max_{p \in \mathbb{R}} \min_{b = \pm 1} [u^\varepsilon(x + \Delta x, t_{j+1}) - p \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| h(x + \Delta x)], \quad (1.2.3)$$

where  $\Delta\hat{x} = \sqrt{2}\varepsilon b$  and  $\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x}) - x$ , associated with the final-time condition

$$u^\varepsilon(x, T) = g(x).$$

Evidently, if  $t_0 = T - K\varepsilon^2$  then

$$u^\varepsilon(x_0, T_0) = \max_{p_0 \in \mathbb{R}} \min_{b_0 = \pm 1} \cdots \max_{p_{K-1} \in \mathbb{R}} \min_{b_{K-1} = \pm 1} \left\{ g(x_K) + \sum_{j=0}^{K-1} -\sqrt{2}\varepsilon b_j p_j + \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j) \right\}, \quad (1.2.4)$$

where  $\Delta\hat{x}_j = \sqrt{2}\varepsilon b_j$  and  $\Delta x_j = \text{proj}_{\overline{\Omega}}(x_j + \Delta\hat{x}_j) - x_j$ . In calling this Helen’s value function, we are using an established convention from the theory of discrete-time, two person games (see, e.g., [69]).

By introducing the operator  $L_\varepsilon$  defined by

$$L_\varepsilon[x, \phi] := \max_{p \in \mathbb{R}} \min_{b = \pm 1} [\phi(x + \Delta x) - p \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| h(x + \Delta x)], \quad (1.2.5)$$

where  $\Delta\hat{x} = \sqrt{2}\varepsilon b$  and  $\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x}) - x$ , the dynamic programming principle (1.2.3) can be written in the form

$$u^\varepsilon(x, t) = L_\varepsilon[x, u^\varepsilon(\cdot, t + \varepsilon^2)]. \quad (1.2.6)$$

We now formally argue that  $u^\varepsilon$  should converge as  $\varepsilon \rightarrow 0$  to the solution of the linear heat equation (1.2.1). The procedure for formal passage from the dynamic programming principle to the associated PDE is familiar: we suppress the dependence of  $u^\varepsilon$  on  $\varepsilon$  and we assume  $u$  is smooth enough to use the Taylor expansion. The first step leads to

$$u(x, t) \approx L_\varepsilon[x, u(\cdot, t + \varepsilon^2)]. \quad (1.2.7)$$

For the second step we need to compute  $L^\varepsilon$  for a  $C^2$ -function  $\phi$ . By the Taylor expansion

$$\begin{aligned} \phi(x + \Delta x) &= \phi(x) + \phi_x(x)\Delta x + \frac{1}{2}\phi_{xx}(x)(\Delta x)^2 + O(\varepsilon^3) \\ &= \phi(x) + \phi_x(x)\Delta \hat{x} + \|\Delta \hat{x} - \Delta x\|\phi_x(x)n(\bar{x}) + \frac{1}{2}\phi_{xx}(x)(\Delta x)^2 + O(\varepsilon^3), \end{aligned}$$

where  $\bar{x} = \text{proj}_{\partial\Omega}(x)$ ,  $\Delta \hat{x} - \Delta x = \|\Delta \hat{x} - \Delta x\|n(\bar{x})$  with  $n$  defined on  $\partial\Omega$  by  $n(x) = 1$  if  $x = c$  and  $n(x) = -1$  if  $x = a$ . Substituting this expression in (1.2.5), we deduce that for all  $C^2$ -function  $\phi$ ,

$$L_\varepsilon[x, \phi] = \phi(x) + \max_{p \in \mathbb{R}} \min_{b = \pm 1} \left[ (\phi_x - p)\Delta \hat{x} + \frac{1}{2}\phi_{xx}(\Delta x)^2 + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - n(\bar{x})\phi_x\} \right] + o(\varepsilon^2). \quad (1.2.8)$$

It remains to compute the max min. If  $\text{dist}(x) > \sqrt{2}\varepsilon$ , we always have  $\Delta x = \Delta \hat{x} = \sqrt{2}\varepsilon b$ , so that the boundary is never crossed and we retrieve the usual situation detailed in [88, Section 2.1]: Helen's optimal choice is  $p = \phi_x$  and  $L_\varepsilon[x, \phi] = \phi(x) + \varepsilon^2 \phi_{xx}(x) + o(\varepsilon^2)$ . If  $\text{dist}(x) < \sqrt{2}\varepsilon$ , we still have  $\Delta \hat{x} = \sqrt{2}b\varepsilon$  but there is a change: if the boundary is crossed,  $\Delta x = \text{dist}(x)$  and  $\|\Delta \hat{x} - \Delta x\| = \sqrt{2}\varepsilon - \text{dist}(x)$ . Suppose that Helen has chosen  $p \in \mathbb{R}$ . Considering the min in (1.2.8), Mark only has two possibilities  $b \in \{\pm 1\}$ . More precisely, suppose that  $x$  is close to  $c$  so that  $\bar{x} = c$  and  $n(\bar{x}) = 1$ ; the case when  $x$  is close to  $a$  is strictly parallel. If Mark chooses  $b = 1$ , the associated value is

$$V_{p,+} = \sqrt{2}(\phi_x - p)\varepsilon + \frac{1}{2}\phi_{xx} \text{dist}^2(x) + (\sqrt{2}\varepsilon - \text{dist}(x))(h(c) - \phi_x),$$

while if Mark chooses  $b = -1$ , the associated value is

$$V_{p,-} = -\sqrt{2}(\phi_x - p)\varepsilon + \phi_{xx}\varepsilon^2.$$

To determine his strategy, Mark compares  $V_{p,-}$  to  $V_{p,+}$ . He chooses  $b = -1$  if  $V_{p,-} < V_{p,+}$ , i.e. if

$$\sqrt{2}(\phi_x - p)\varepsilon + \frac{1}{2}\phi_{xx} \text{dist}^2(x) + (\sqrt{2}\varepsilon - \text{dist}(x))(h(c) - \phi_x) > -\sqrt{2}(\phi_x - p)\varepsilon + \phi_{xx}\varepsilon^2,$$

that we can rearrange into

$$2\sqrt{2}(\phi_x - p)\varepsilon > \phi_{xx} \left( \varepsilon^2 - \frac{\text{dist}^2(x)}{2} \right) - (\sqrt{2}\varepsilon - \text{dist}(x)) [h(c) - \phi_x].$$

This last inequality yields an explicit condition on the choice of  $p$  previously made by Helen

$$p < p_{\text{opt}} := \phi_x + \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\sqrt{2}\varepsilon} \right) [h(c) - \phi_x] + \frac{1}{2\sqrt{2}} \phi_{xx} \left( 1 - \frac{\text{dist}^2(x)}{2\varepsilon^2} \right) \varepsilon. \quad (1.2.9)$$

Meanwhile Mark chooses  $b = 1$  if  $V_{p,+} < V_{p,-}$ , which leads to the reverse inequality  $p > p_{\text{opt}}$ . The situation when  $V_{p,+} = V_{p,-}$  obviously corresponds to  $p = p_{\text{opt}}$ . We deduce that

$$L_\varepsilon[x, \phi] = \max \left[ \max_{p \leq p_{\text{opt}}} V_{p,-}, V_{p_{\text{opt}},-}, \max_{p \geq p_{\text{opt}}} V_{p,+} \right].$$

Helen wants to optimize her choice of  $p$ . The functions  $V_{p,+}$  and  $V_{p,-}$  are both affine on  $\phi_x - p$ . The first one is decreasing while the second is increasing with respect to  $p$ . As a result, we deduce that Helen's

optimal choice is  $p = p_{\text{opt}}$  as defined in (1.2.9) and  $L_\varepsilon[x, \phi] = V_{p_{\text{opt}},+} = V_{p_{\text{opt}},-}$ . We notice that Helen behaves optimally by becoming indifferent to Mark's choice; our games will not always conserve this feature, which was observed in [88]. Finally, for all  $C^2$ -function  $\phi$ , we have

$$\begin{aligned} L_\varepsilon[x, \phi] &= \phi(x) \\ &+ \begin{cases} \frac{\varepsilon}{\sqrt{2}} \left(1 - \frac{\text{dist}(x)}{\sqrt{2}\varepsilon}\right) [h(\bar{x}) - n(\bar{x})\phi_x(x)] + \frac{\varepsilon^2}{2} \phi_{xx}(x) \left(1 + \frac{\text{dist}^2(x)}{2\varepsilon^2}\right) + o(\varepsilon^2), & \text{if } \text{dist}(x) \leq \sqrt{2}\varepsilon, \\ \varepsilon^2 \phi_{xx}(x) + o(\varepsilon^2), & \text{if } \text{dist}(x) \geq \sqrt{2}\varepsilon. \end{cases} \end{aligned} \quad (1.2.10)$$

Since  $u$  is supposed to be smooth, the Taylor expansion on  $t$  yields that  $u(\cdot, t + \varepsilon^2) = u(\cdot, t) + u_t(\cdot, t)\varepsilon^2 + o(\varepsilon^2)$  and we formally derive the PDE by plugging (1.2.10) in (1.2.7). This gives

$$0 \approx \varepsilon^2 u_t + \begin{cases} \frac{\varepsilon}{\sqrt{2}} \left(1 - \frac{\text{dist}(x)}{\sqrt{2}\varepsilon}\right) [h(\bar{x}) - n(\bar{x})u_x] + \frac{\varepsilon^2}{2} u_{xx} \left(1 + \frac{\text{dist}^2(x)}{2\varepsilon^2}\right) + o(\varepsilon^2), & \text{if } \text{dist}(x) \leq \sqrt{2}\varepsilon, \\ \varepsilon^2 u_{xx} + o(\varepsilon^2), & \text{if } \text{dist}(x) \geq \sqrt{2}\varepsilon. \end{cases} \quad (1.2.11)$$

If  $x \in \Omega$ , for  $\varepsilon$  small enough, the second alternative in (1.2.11) is always valid so that we deduce from the  $\varepsilon^2$ -order terms in (1.2.11) that  $u_t + u_{xx} = 0$ . If  $x$  is on the boundary  $\partial\Omega$ , then  $\text{dist}(x) = 0$ ,  $\bar{x} = x$  and the first possibility in (1.2.11) is always satisfied. We observe that the  $\varepsilon$ -order term is predominant since  $\varepsilon \gg \varepsilon^2$ . By dividing by  $\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we obtain  $h(x) - u_x(x) \cdot n(x) = 0$ .

Now we present a financial interpretation of this game. Helen plays the role of a hedger or an investor, while Mark represents the market. The position  $x$  is a stock price which evolves in  $\bar{\Omega}$  as a function of time  $t$ , starting at  $x_0$  at time  $t_0$  and the boundary  $\partial\Omega$  plays the role of barriers which additionally determine a coupon when the stock price crosses  $\partial\Omega$ . The small parameter  $\varepsilon$  determines both the stock price increments  $\Delta\hat{x} \leq \sqrt{2}\varepsilon$  and the time step  $\varepsilon^2$ . Helen's score keeps track of the profits and losses generated by her hedging activity.

Helen's situation is as follows: she holds an option that will pay her  $g(x(T))$  at time  $T$  ( $g$  could be negative). Her goal is to hedge this position by buying or selling the stock at each time increment. She can borrow and lend money without paying or collecting any interest, and can take any (long or short) stock position she desires. At each step, Helen chooses a real number  $p_j$  (depending on  $x_j$  and  $t_j$ ), then adjusts her portfolio so it contains  $-p_j$  units of stock (borrowing or lending to finance the transaction, so there is no change in her overall wealth). Mark sees Helen's choice. Taking it into account, he makes the stock go up or down (i.e. he chooses  $b_j = \pm 1$ ), trying to degrade her outcome. The stock price changes from  $x_j$  to  $x_{j+1} = \text{proj}_{\bar{\Omega}}(x_j + \Delta\hat{x}_j)$ , and Helen's wealth changes by  $-\sqrt{2}\varepsilon b_j p_j + \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j)$  (she has a profit if it is positive, a loss if it is negative). The term  $\|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j)$  is a coupon that will be produced only if the special event  $\Delta\hat{x}_j \notin \Omega$  happens. The hedger must take into account the possibility of this new event. The hedging parameter  $p_j$  is modified close to the boundary but the hedger's value function is still independent from the variations of the market. At the final time Helen collects her option payoff  $g(x_K)$ . If Helen and Mark both behave optimally at each stage, then we deduce by (1.2.4) that

$$u^\varepsilon(x_0, t_0) + \sum_{j=0}^{K-1} \sqrt{2}\varepsilon b_j p_j - \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j) = g(x_K).$$

Helen's decisions are in fact identical to those of an investor hedging an option with payoff  $g(x)$  and coupon  $h(x)$  if the underlying asset crosses the barrier  $\partial\Omega$  in a binomial-tree market with  $\Delta\hat{x} = \sqrt{2}\varepsilon$  at each timestep.

## 1.2.2 General parabolic equations

This section explains what to do when  $\mathcal{F}$  depends on  $Du$ ,  $D^2u$  and also on  $u$ . We also permit dependence on  $x$  and  $t$ , so we are now discussing a fully-nonlinear (degenerate) parabolic equation of the

form

$$\begin{cases} \partial_t u - \mathcal{F}(t, x, u, Du, D^2u) = 0, & \text{for } x \in \Omega \text{ and } t < T, \\ \langle Du(x, t), n(x) \rangle = h(x), & \text{for } x \in \partial\Omega \text{ and } t < T, \\ u(x, T) = g(x), & \text{for } x \in \overline{\Omega}, \end{cases} \quad (1.2.12)$$

where  $\Omega$  is a  $C^2$ -domain satisfying both the uniform interior and exterior ball conditions and the boundary condition  $h$  and the final-time data  $g$  are uniformly bounded, continuous, depending only on  $x$ .

There are two players, Helen and Mark; a small parameter  $\varepsilon$  is fixed. Since the PDE is to be solved in  $\Omega$ , Helen's final-time bonus  $g$  is now a function of  $x \in \overline{\Omega}$  and Helen's coupon profile  $h$  is a function of  $x \in \partial\Omega$ . The state of the game is described by its spatial position  $x \in \overline{\Omega}$  and Helen's debt  $z \in \mathbb{R}$ . Helen's goal is to minimize her final debt, while Mark's is to obstruct her.

The rules of the game depend on three new parameters,  $\alpha, \beta, \gamma > 0$  whose presence represents no loss of generality. Their role will be clear in a moment. The requirements

$$\alpha < 1/3, \quad (1.2.13)$$

and

$$\alpha + \beta < 1, \quad 2\alpha + \gamma < 2, \quad \max(\beta q, \beta r) < 2, \quad (1.2.14)$$

will be clear in the explanation of the game. However, the proof of convergence in Section 1.3 and consistency in Section 1.4 needs more: there we will require

$$\gamma < 1 - \alpha, \quad \beta(q - 1) < \alpha + 1, \quad \gamma(r - 1) < 2\alpha, \quad \gamma r < 1 + \alpha. \quad (1.2.15)$$

These conditions do not restrict the class of PDEs we consider, since for any  $q$  and  $r$  there exist  $\alpha, \beta$  and  $\gamma$  with the desired properties.

Using the language of our financial interpretation:

- a) First we consider  $U^\varepsilon(x, z, t)$ , Helen's optimal wealth at time  $T$ , if initially at time  $t$  the stock price is  $x$  and her wealth is  $-z$ .
- b) Then we define  $u^\varepsilon(x, t)$  or  $v^\varepsilon(x, t)$  as, roughly speaking, the initial debt Helen should have at time  $t$  to break even at time  $T$ .

The proper definition of  $U^\varepsilon(x, z, t)$  involves a game similar to that of Section 1.2.1. The rules are as follows: if at time  $t_j = t_0 + j\varepsilon^2$ , the position is  $x_j$  and Helen's debt is  $z_j$ , then

**Step 1:** Helen chooses a vector  $p_j \in \mathbb{R}^d$  and a matrix  $\Gamma_j \in \mathcal{S}_d$ , restricted by

$$\|p_j\| \leq \varepsilon^{-\beta}, \quad \|\Gamma_j\| \leq \varepsilon^{-\gamma}. \quad (1.2.16)$$

**Step 2:** Taking Helen's choice into account, Mark chooses the stock price  $x_{j+1}$  so as to degrade Helen's outcome. Mark chooses an intermediate point  $\hat{x}_{j+1} = x_j + \Delta\hat{x}_j \in \mathbb{R}^d$  such that

$$\|\Delta\hat{x}_j\| \leq \varepsilon^{1-\alpha}. \quad (1.2.17)$$

This position  $\hat{x}_{j+1}$  determines the new position  $x_{j+1} = x_j + \Delta x_j \in \overline{\Omega}$  at time  $t_{j+1}$  by the rule

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}). \quad (1.2.18)$$

**Step 3:** Helen's debt changes to

$$z_{j+1} = z_j + p_j \cdot \Delta\hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta\hat{x}_j, \Delta\hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(t_j, x_j, z_j, p_j, \Gamma_j) - \|\Delta\hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j). \quad (1.2.19)$$

**Step 4:** The clock steps forward to  $t_{j+1} = t_j + \varepsilon^2$  and the process repeats, stopping when  $t_K = T$ . At the final time Helen receives  $g(x_K)$  from the option.

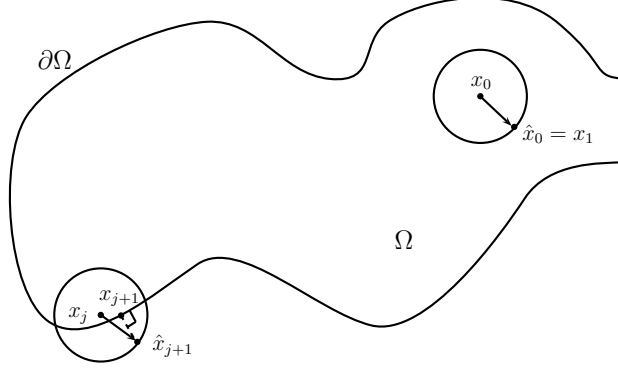


Figure 1.1: Rules of the game, admissible moves near the boundary and inside the domain.

This game is well-posed for all  $\varepsilon > 0$  small enough. As mentioned in the introduction, the uniform exterior ball condition holds automatically for a  $C^2$ -bounded domain. In this case, by compactness of  $\partial\Omega$ , there exists  $\varepsilon_* > 0$  such that  $\text{proj}_{\overline{\Omega}}$  is well-defined for all  $x \in \mathbb{R}^d$  such that  $\text{dist}(x, \overline{\Omega}) \leq \varepsilon_*$ . It can be noticed that an unbounded  $C^2$ -domain, even with bounded curvature, does not generally satisfy this condition. Since the domain  $\Omega$  satisfies the uniform exterior ball condition given by Definition 1.1.2 for a certain  $r$ , the projection is well-defined on the tubular neighborhood  $\{x \in \mathbb{R}^d \setminus \overline{\Omega} : \text{dist}(x) < r/2\}$  of the boundary.

**Remark 1.2.2.** To give a sense to (1.2.19) for all  $\Delta x_j$ , the function  $h$  which is defined only on the boundary can be extended on  $\overline{\Omega}$  by any function  $\Omega \rightarrow \mathbb{R}$  since  $\|x_{j+1} - \hat{x}_{j+1}\|$  is different from zero if and only if  $\hat{x}_{j+1} \notin \overline{\Omega}$ . Moreover, by comparing  $\Delta \hat{x}_j$  and  $\Delta x_j$ , one gets the relation

$$x_{j+1} = \hat{x}_{j+1} + \Delta x_j - \Delta \hat{x}_j.$$

If  $\hat{x}_{j+1} \in \Omega$ , then  $x_{j+1} = \hat{x}_{j+1}$  and the rules of the usual game [88] are retrieved. Figure 1.1 presents the two geometric situations for the choice for Mark:  $B(x, \varepsilon^{1-\alpha}) \subset \Omega$  or not.

Helen's goal is to maximize her worst-case score at time  $T$ , and Mark's is to work against her. Her value function is

$$U^\varepsilon(x_0, z_0, t_0) = \max_{\text{Helen's choices}} [g(x_K) - z_K].$$

It is characterized by the dynamic programming principle

$$U^\varepsilon(x, z, t_j) = \max_{p, \Gamma} \min_{\Delta \hat{x}} U^\varepsilon(x + \Delta x, z + \Delta z, t_{j+1}) \quad (1.2.20)$$

together with the final-time condition  $U^\varepsilon(x, z, T) = g(x) - z$ . Here  $\Delta \hat{x}$  is  $\hat{x}_{j+1} - x_j$ ,  $\Delta x$  is determined by

$$\Delta x = x_{j+1} - x_j = \text{proj}_{\overline{\Omega}}(x_j + \Delta \hat{x}_j) - x_j, \quad (1.2.21)$$

and  $\Delta z = z_{j+1} - z_j$  is given by (1.2.19), and the optimizations are constrained by (1.2.16) and (1.2.17). It is easy to see that the max/min is achieved and is a continuous function of  $x$  and  $z$  at each discrete time (the proof is by induction backward in time, like the argument sketched in [88]).

When  $\mathcal{F}$  depends on  $z$ , the function  $z \mapsto U^\varepsilon(x, z, t)$  can be nonmonotone, so we must distinguish between the minimal and maximal debt with which Helen breaks even at time  $T$ . Thus, following [42], we define

$$u^\varepsilon(x_0, t_0) = \sup\{z_0 : U^\varepsilon(x_0, z_0, t_0) \geq 0\} \quad (1.2.22)$$

and

$$v^\varepsilon(x_0, t_0) = \inf\{z_0 : U^\varepsilon(x_0, z_0, t_0) \leq 0\}, \quad (1.2.23)$$

with the convention that the empty set has  $\sup = -\infty$  and  $\inf = \infty$ . Clearly  $v^\varepsilon \leq u^\varepsilon$ , and  $u^\varepsilon(x, T) = v^\varepsilon(x, T) = g(x)$ . Since the definitions of  $u^\varepsilon$  and  $v^\varepsilon$  are implicit, these functions cannot be characterized by a dynamic programming principle. However we still have two “dynamic programming inequalities”.

**Proposition 1.2.3.** *If  $u^\varepsilon(x, t)$  is finite then*

$$u^\varepsilon(x, t) \leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ u^\varepsilon(x + \Delta x, t + \varepsilon^2) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, u^\varepsilon(x, t), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right]. \quad (1.2.24)$$

Similarly, if  $v^\varepsilon(x, t)$  is finite then

$$v^\varepsilon(x, t) \geq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ v^\varepsilon(x + \Delta x, t + \varepsilon^2) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, v^\varepsilon(x, t), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right]. \quad (1.2.25)$$

The sup and inf are constrained by (1.2.16) and (1.2.17) and  $\Delta x$  is determined by (1.2.21).

*Proof.* The argument follows the same lines as the proof of the dynamic programming inequalities given in [88, Proposition 2.1]. For sake of completeness we give here the details. To prove (1.2.24), consider  $z = u^\varepsilon(x, t)$ . By the definition of  $u^\varepsilon$  (and remembering that  $U^\varepsilon$  is continuous) we have  $U^\varepsilon(x, z, t) = 0$ . Hence writing (1.2.20), we have

$$0 = \max_{p, \Gamma} \min_{\Delta \hat{x}} U^\varepsilon \left( x + \Delta x, z + p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x), t + \varepsilon^2 \right).$$

We conclude that there exist  $p, \Gamma$  (constrained by (1.2.16)) such that for all  $\Delta \hat{x}$  constrained by (1.2.17), determining  $\Delta x$  by (1.2.21), we have

$$U^\varepsilon \left( x + \Delta x, z + p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x), t + \varepsilon^2 \right) \geq 0.$$

By the definition of  $u^\varepsilon$  given by (1.2.22), this implies that

$$z + p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \leq u^\varepsilon(x + \Delta x, t + \varepsilon^2).$$

In other words, there exist  $p, \Gamma$  such that for every  $\Delta \hat{x}$ , determining  $\Delta x$  by (1.2.21),

$$z \leq u^\varepsilon(x + \Delta x, t + \varepsilon^2) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right).$$

Recalling that  $z = u^\varepsilon(x, t)$  and passing to the inf and sup, we get (1.2.24). The proof of (1.2.25) follows exactly the same lines.  $\square$

To define viscosity subsolutions and supersolutions, we shall follow the Barles and Perthame procedure [18], let us recall the upper and lower relaxed semi-limits defined for  $(t, x) \in [0, T] \times \overline{\Omega}$  as

$$\bar{u}(x, t) = \limsup_{\substack{y \rightarrow x, y \in \overline{\Omega} \\ t_j \rightarrow t \\ \varepsilon \rightarrow 0}} u^\varepsilon(y, t_j) \quad \text{and} \quad \underline{v}(x, t) = \liminf_{\substack{y \rightarrow x, y \in \overline{\Omega} \\ t_j \rightarrow t \\ \varepsilon \rightarrow 0}} v^\varepsilon(y, t_j), \quad (1.2.26)$$

where the discrete times are  $t_j = T - j\varepsilon^2$ . We shall show, under suitable hypotheses, that  $\underline{v}$  and  $\bar{u}$  are respectively viscosity super and subsolutions of (1.2.12). Before stating our rigorous result in Section 1.2.2.2, the next section presents the heuristic derivation of the PDE (1.2.12) through the optimal strategies of Helen and Mark.

### 1.2.2.1 Heuristic derivation of the optimal player strategies

We now formally show that  $u^\varepsilon$  should converge as  $\varepsilon \rightarrow 0$  to the solution of (1.2.12). Roughly speaking, the PDE (1.2.12) is the formal Hamilton-Jacobi-Bellman equation associated to the two-persons game presented at the beginning of the present section. The procedure for formal derivation from the dynamic programming principle to a corresponding PDE is classical: we assume  $u^\varepsilon$  and  $v^\varepsilon$  coincide and are smooth to use Taylor expansion, suppress the dependence of  $u^\varepsilon$  and  $v^\varepsilon$  on  $\varepsilon$  and finally make  $\varepsilon \rightarrow 0$ . That has already been done for  $x$  far from the boundary in [88, Section 2.2] for  $\mathcal{F}$  depending only on  $(Du, D^2u)$ . We now suppose that  $x$  is close enough to the boundary so that  $\hat{x}$  can be nontrivial. By assuming  $u^\varepsilon = v^\varepsilon$  as announced and suppressing the dependence of  $u^\varepsilon$  on  $\varepsilon$ , the two dynamic programming inequalities (1.2.24) and (1.2.25) give the dynamic programming equality

$$u(x, t) \approx \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ u(x + \Delta x, t + \varepsilon^2) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, u(x, t), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right]. \quad (1.2.27)$$

Remembering that  $\Delta \hat{x}$  is small, if  $u$  is assumed to be smooth, we obtain

$$\begin{aligned} u(x + \Delta x, t + \varepsilon^2) + \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \\ \approx u(x, t) + \varepsilon^2 u_t + Du \cdot \Delta x + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle + \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \\ \approx u(x, t) + \varepsilon^2 u_t + Du \cdot \Delta \hat{x} + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - Du \cdot n(x + \Delta x)\} + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle, \end{aligned}$$

since the outer normal can be expressed by  $n(x + \Delta x) = -\frac{\Delta x - \Delta \hat{x}}{\|\Delta \hat{x} - \Delta x\|}$  if  $\hat{x} \notin \Omega$ . Substituting this computation in (1.2.27), and rearranging the terms, we get

$$\begin{aligned} 0 \approx \varepsilon^2 u_t + \max_{p, \Gamma} \min_{\Delta \hat{x}} [(Du - p) \cdot \Delta \hat{x} + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - Du \cdot n(x + \Delta x)\} \\ + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, u, p, \Gamma)] \quad (1.2.28) \end{aligned}$$

where  $u$ ,  $Du$ ,  $D^2u$  are evaluated at  $(x, t)$ . We have ignored the upper bounds in (1.2.16) since they allow  $p$ ,  $\Gamma$  to be arbitrarily large in the limit  $\varepsilon \rightarrow 0$  (we shall of course be more careful in Section 1.4).

If the domain  $\Omega$  does not satisfy the uniform interior ball condition,  $\Omega$  can present an infinity number of “neck pitchings” of neck size arbitrarily small. To avoid this situation, the uniform interior ball condition is used to impose a strictly positive lower bound on these necks. If  $x$  is supposed to be extremely close to the  $C^2$ -boundary and  $\|\Delta \hat{x}\| \leq \varepsilon^{1-\alpha}$ , the boundary looks like a hyperplane orthogonal to the outer normal vector  $n(\bar{x})$ , where  $\bar{x}$  is the projection of  $x$  on the boundary  $\partial\Omega$  (see Figure 1.2). By Gram-Schmidt process, we can find some vectors  $e_2, \dots, e_d$  such that  $(e_1 = n(\bar{x}), e_2, \dots, e_d)$  form an orthonormal basis of  $\mathbb{R}^d$ . In this basis, denote

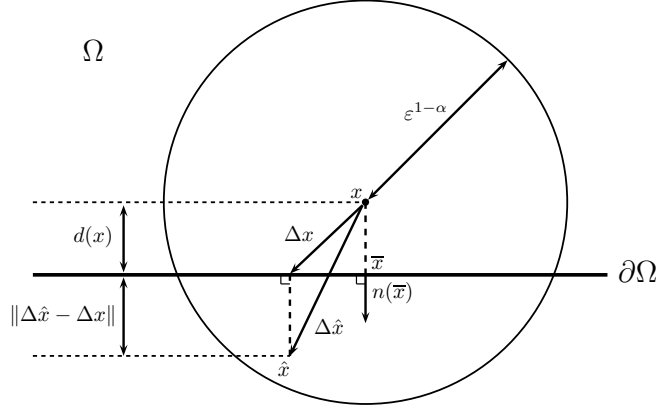
$$p = p_1 n(\bar{x}) + \tilde{p} \quad \text{and} \quad \Gamma = (\langle \Gamma e_i, e_j \rangle)_{1 \leq i, j \leq d} = \left( \begin{array}{c|ccc} \Gamma_{11} & \cdots & (\Gamma_{1i})_{2 \leq i \leq d} & \cdots \\ \vdots & & & \\ (\Gamma_{i1})_{2 \leq i \leq N} & & \tilde{\Gamma} & \\ \vdots & & & \end{array} \right), \quad (1.2.29)$$

where  $p_1 \in \mathbb{R}$ ,  $\tilde{p} \in V^\perp = \text{span}(e_2, \dots, e_d)$  and  $\tilde{\Gamma} = (\langle \Gamma e_i, e_j \rangle)_{2 \leq i, j \leq d} \in \mathcal{S}_{d-1}$ .

Let us focus on the Neumann penalization term in (1.2.28) denoted by

$$P(x) = \|\Delta \hat{x} - \Delta x\| m(\Delta x) \quad \text{with} \quad m(\Delta x) = \begin{cases} h(x + \Delta x) - Du(x) \cdot n(x + \Delta x), & \text{if } \hat{x} \notin \bar{\Omega}, \\ \tilde{m}(\Delta x), & \text{if } \hat{x} \in \bar{\Omega}, \end{cases}$$




 Figure 1.2: Formal derivation for  $x$  near the boundary  $\partial\Omega$ , notation:  $\bar{x} = \text{proj}_{\partial\Omega}(x)$ .

where  $m(\Delta x)$  is extended for  $\hat{x} \in \bar{\Omega}$  by any function  $\tilde{m}(\Delta x)$  (see Remark 1.2.2). This contribution is favorable to Helen,  $P(x) > 0$ , if  $m(x) > 0$ , or to Mark,  $P(x) < 0$ , if  $m(x) < 0$ , and its size depends on the magnitude of the vector  $\Delta\hat{x} - \Delta x$ . Our formal derivation is local and essentially geometric, in the sense that our target is to determine the optimal choices for Helen by considering all the moves  $\Delta\hat{x}$  that Mark can choose. By continuity of  $h$  and smoothness of  $u$ , the function  $m(\Delta x)$  is close to

$$m = h(\bar{x}) - Du(x) \cdot n(\bar{x}) \quad \text{if } \hat{x} \notin \bar{\Omega}. \quad (1.2.30)$$

We shall assume here that  $m(\Delta x)$ , which serves to model the Neumann boundary condition, is locally constant on the boundary and equal to  $m$ . This hypothesis corresponds in the game to assume that in a small neighborhood, crossing the boundary is always favorable to one player. In order to focus only on the geometric aspects, this approach seems formally appropriate since it freezes the dependence of  $p(x)$  on  $m(x)$  by eliminating the difficulties linked to the local variations of  $m(x)$  like the change of sign. Hence, it is sufficient to examine

$$\max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (Du - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| m + \frac{1}{2} \langle D^2 u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, u, p, \Gamma) \right], \quad (1.2.31)$$

where  $m$  is given by (1.2.30).

The formal proof will be performed in three steps.

**Step 1:** To determine the optimal choice for Helen of  $p$ , we consider the  $\varepsilon$ -order optimization problem  $\mathcal{M}$  obtained from (1.2.31) by neglecting the second  $\varepsilon$ -order terms

$$\mathcal{M} := \max_p \min_{\Delta\hat{x}} [(Du - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| m]. \quad (1.2.32)$$

By writing  $\Delta\hat{x} = (\Delta\hat{x})_1 n(\bar{x}) + \widetilde{\Delta\hat{x}}$  with  $\widetilde{\Delta\hat{x}} \in V^\perp$  and observing that  $\|\Delta\hat{x} - \Delta x\|$  depends only on  $(\Delta\hat{x})_1$ , we decompose the max min (1.2.32) into

$$\begin{aligned} \mathcal{M} &= \max_{p_1, \tilde{p}} \min_{\Delta\hat{x}} \left[ (\widetilde{Du} - \tilde{p}) \cdot \widetilde{\Delta\hat{x}} + (Du_1 - p_1)(\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| \right] \\ &= \max_{p_1} \min_{|(\Delta\hat{x})_1| \leq \varepsilon^{1-\alpha}} \left[ (Du_1 - p_1)(\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| m + \max_{\tilde{p}} \min_{\|\widetilde{\Delta\hat{x}}\| \leq \sqrt{\varepsilon^{2-2\alpha} - |(\Delta\hat{x})_1|^2}} (\widetilde{Du} - \tilde{p}) \cdot \widetilde{\Delta\hat{x}} \right]. \end{aligned}$$

Noticing that the choices of  $\tilde{p}$  and  $p_1$  are independent from each other, we can successively solve the optimization problems. First of all, in order to choose  $\tilde{p}$ , let us determine

$$\widetilde{\mathcal{M}} = \max_{\tilde{p}} \min_{\|\widetilde{\Delta\hat{x}}\| \leq \sqrt{\varepsilon^{2-2\alpha} - |(\Delta\hat{x})_1|^2}} (\widetilde{Du} - \tilde{p}) \cdot \widetilde{\Delta\hat{x}}.$$

For the particular moves  $\Delta\hat{x} = \pm\varepsilon^{1-\alpha}n(\bar{x})$ ,  $\widetilde{\Delta\hat{x}} = 0$ , the min above is always zero and Helen's choice of  $\tilde{p}$  is indifferent. Moving these moves aside, Helen should take  $\tilde{p} = \text{proj}_{V^\perp} Du = \widetilde{Du}$ , since otherwise Mark can make this max min strictly negative and minimal by choosing  $\widetilde{\Delta\hat{x}} = -\sqrt{\varepsilon^{2-2\alpha} - |(\Delta\hat{x})_1|^2} \frac{(Du-p)_{V^\perp}}{\|Du-p\|}$  with  $\Delta\hat{x} \neq \pm\varepsilon^{1-\alpha}n(\bar{x})$ . Thus Helen chooses  $\tilde{p} = \widetilde{Du}$ ,  $\widetilde{\mathcal{M}} = 0$  and  $\mathcal{M}$  reduces to

$$\mathcal{M} = \max_{p_1} \min_{\Delta\hat{x}} [((Du)_1 - p_1)(\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\|m]. \quad (1.2.33)$$

To determine the remaining coordinate  $p_1 = p \cdot n(\bar{x})$  of  $p$ , we now consider the optimization problem (1.2.33) by restricting the possible choices made by Mark to the moves  $\Delta\hat{x}$  which belong to the subspace  $V = \mathbb{R}n(\bar{x})$ . Since  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$  and  $\Delta\hat{x} \in V$ , we use the parametrization  $\Delta\hat{x} = \lambda\varepsilon^{1-\alpha}n(\bar{x})$ ,  $\lambda \in [-1, 1]$ . If  $\hat{x} \in \Omega$ , the boundary is not crossed and  $\|\Delta x - \Delta\hat{x}\| = 0$ , while if  $\hat{x} \notin \Omega$  the boundary is crossed and  $\|\Delta x - \Delta\hat{x}\| = \lambda\varepsilon^{1-\alpha} - \text{dist}(x)$ . The intermediate point  $\hat{x} = \bar{x} \in \partial\Omega$  separating the two regions corresponds to  $\lambda_0 = \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}$  and  $\|\Delta x - \Delta\hat{x}\| = 0$ . As a result, to compute the min in (1.2.33), we shall distinguish these two regions by decomposing the global minimization problem into two minimization problems respectively on each region

$$\mathcal{M} = \max_{s_p} \kappa(s_p) \quad \text{with} \quad \kappa(s_p) := \min(\mathcal{M}_1(s_p), \mathcal{M}_2(s_p)), \quad (1.2.34)$$

where  $s_p := (Du - p) \cdot n(\bar{x})$  and

$$\mathcal{M}_1(s_p) := \min_{\lambda_0 \leq \lambda_1 \leq 1} M_1(\lambda_1) \quad \text{with} \quad M_1(\lambda_1) := (s_p + m)\varepsilon^{1-\alpha}\lambda_1 - \text{dist}(x)m, \quad (1.2.35)$$

$$\mathcal{M}_2(s_p) := \min_{-1 \leq \lambda_2 \leq \lambda_0} M_2(\lambda_2) \quad \text{with} \quad M_2(\lambda_2) := s_p\varepsilon^{1-\alpha}\lambda_2. \quad (1.2.36)$$

For fixed  $p$ , the functions defining  $M_1$  and  $M_2$  are affine and can easily be minimized separately:

- If  $s_p + m \geq 0$ ,  $\mathcal{M}_1(s_p)$  is attained for  $\lambda_1 = \lambda_0$  and  $\mathcal{M}_1(s_p) = \text{dist}(x)s_p$ .
- If  $s_p + m < 0$ ,  $\mathcal{M}_1(s_p)$  is attained for  $\lambda_1 = 1$  and  $\mathcal{M}_1(s_p) = \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m$ .
- If  $s_p \geq 0$ ,  $\mathcal{M}_2(s_p)$  is attained for  $\lambda_2 = -1$  and  $\mathcal{M}_2(s_p) = -\varepsilon^{1-\alpha}s_p$ .
- If  $s_p < 0$ ,  $\mathcal{M}_2(s_p)$  is attained for  $\lambda_2 = \lambda_0$  and  $\mathcal{M}_2(s_p) = \text{dist}(x)s_p$ .

Geometrically,  $\lambda \in \{-1, 1, \lambda_0\}$  corresponds to three particular moves:  $\Delta\hat{x} = \pm\varepsilon^{1-\alpha}n(\bar{x})$  and  $\Delta\hat{x} = \text{dist}(x)n(\bar{x})$ . We are going to distinguish several cases to compute the max min according to the sign of  $s_p$  and  $m$ . First of all, let us assume that  $m$  is positive.

- (C1) If  $s_p \geq 0$  then  $s_p + m \geq 0$  and the optimal choices are  $(\lambda_1, \lambda_2) = (\lambda_0, -1)$ . It remains to minimize between (1.2.35) and (1.2.36). Taking into account that  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $s_p \geq 0$ , we get by the definition of  $\kappa(s_p)$  given by (1.2.34) that  $\kappa(s_p) = \min\{\text{dist}(x)s_p, -\varepsilon^{1-\alpha}s_p\} = -\varepsilon^{1-\alpha}s_p$ .
- (C2) If  $-m \leq s_p < 0$  then  $(\lambda_1, \lambda_2) = (\lambda_0, \lambda_0)$  and  $\kappa(s_p) = \mathcal{M}_1(s_p) = \mathcal{M}_2(s_p) = \text{dist}(x)s_p$ .
- (C3) If  $s_p < -m < 0$  then  $(\lambda_1, \lambda_2) = (1, \lambda_0)$  and  $\mathcal{M}_1(s_p) = \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m$  and  $\mathcal{M}_2(s_p) = \text{dist}(x)s_p$ . By multiplying the inequality  $s_p < -m < 0$  by  $(\varepsilon^{1-\alpha} - \text{dist}(x))$ , we get

$$\kappa(s_p) = \min\{\varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m, \text{dist}(x)s_p\} = \text{dist}(x)s_p.$$

By combining cases (C1)–(C3), we conclude that if  $m > 0$ ,

$$\kappa(s_p) = \begin{cases} \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m, & \text{if } s_p \leq -m, \\ \text{dist}(x)s_p, & \text{if } -m \leq s_p \leq 0, \\ -\varepsilon^{1-\alpha}s_p, & \text{if } s_p \geq 0. \end{cases}$$

The max of  $\kappa$  is zero and reached at the unique value  $s_p = Du \cdot n(\bar{x}) - p_1 = 0$ . Since  $\tilde{p} = \widetilde{Du}$  by the previous analysis, we conclude in (1.2.29) that if  $m > 0$ , Helen's optimal choice is  $p = Du$ .

Let us now suppose that  $m$  is negative.

- (C4) If  $s_p < 0$  then  $s_p + m < 0$  and the optimal choices are  $(\lambda_1, \lambda_2) = (1, \lambda_0)$ . By the definition of  $\kappa(s_p)$  given by (1.2.34), we obtain

$$\kappa(s_p) = \min\{\varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m, \text{dist}(x)s_p\} = \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m. \quad (1.2.37)$$

- (C5) If  $s_p \geq -m > 0$  then  $(\lambda_1, \lambda_2) = (\lambda_0, -1)$  and  $\mathcal{M}_1(s_p) = \text{dist}(x)s_p$  and  $\mathcal{M}_2(s_p) = -\varepsilon^{1-\alpha}s_p$ . By the definition of  $\kappa(s_p)$  given by (1.2.34), we obtain  $\kappa(s_p) = \min\{\text{dist}(x)s_p, -\varepsilon^{1-\alpha}s_p\} = -\varepsilon^{1-\alpha}s_p$ .
- (C6) If  $0 < s_p < -m$ , then  $(\lambda_1, \lambda_2) = (1, -1)$  and  $\mathcal{M}_1(s_p) = \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m$  and  $\mathcal{M}_2(s_p) = -\varepsilon^{1-\alpha}s_p$ . By the definition of  $\kappa(s_p)$  given by (1.2.34), we obtain

$$\kappa(s_p) = \min\{\varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m, -\varepsilon^{1-\alpha}s_p\}.$$

The target for Helen is to maximize this minimum with respect to  $s_p$ . Both functions intervening in the minimum are affine: the first one is affine, strictly increasing and is equal to  $(\varepsilon^{1-\alpha} - \text{dist}(x))m < 0$  for  $s_p = 0$  and to  $\text{dist}(x)m > 0$  for  $s_p = -m$  whereas the second function is linear and strictly decreasing and is equal to  $m\varepsilon^{1-\alpha} < 0$  for  $s_p = -m$ . As a result, there is a unique  $s^*$  such that these two functions are equal and this value precisely realizes the max of  $\kappa$  on  $[0, -m]$ . Thus, the best that Helen can hope corresponds to  $\varepsilon^{1-\alpha}s^* + (\varepsilon^{1-\alpha} - \text{dist}(x))m = -\varepsilon^{1-\alpha}s^*$ . This gives

$$s^* = (Du - p) \cdot n(\bar{x}) = -\frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m.$$

We immediately check that  $s^* \in [0, -\frac{m}{2}]$ , which implies the condition  $s^* + m \leq \frac{1}{2}m < 0$ . Thus,

$$\max_{s_p \in [0, -m]} \kappa(s_p) = \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m \text{ is greater than the minimum obtained in (1.2.37).}$$

By combining cases (C4)–(C6), we conclude that if  $m \leq 0$ ,

$$\kappa(s_p) = \begin{cases} \varepsilon^{1-\alpha}s_p + (\varepsilon^{1-\alpha} - \text{dist}(x))m, & \text{if } s_p < s^*, \\ -\varepsilon^{1-\alpha}s_p, & \text{if } s_p \geq s^*. \end{cases}$$

The max of  $\kappa$  is equal to  $\kappa(s^*)$  and reached for  $s_p = Du \cdot n(\bar{x}) - p_1 = s^*$ .

Let us give an intermediate conclusion: if  $m > 0$ , Helen chooses  $p = Du$  whereas if  $m \leq 0$ , she chooses

$$p = Du + \frac{m}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) n(\bar{x}). \quad (1.2.38)$$

**Step 2:** We are now going to take into account the second order terms in  $\varepsilon$  in the optimization problem. If  $m \geq 0$ , once Helen has chosen  $p = Du$ , the optimization problem (1.2.31) reduces to computing

$$\max_{\Gamma} \min_{\Delta \hat{x}} \left[ \|\Delta \hat{x} - \Delta x\|m + \frac{1}{2} \langle D^2u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, u, Du, \Gamma) \right]. \quad (1.2.39)$$

Mark is going to choose  $\Delta \hat{x} \cdot n(\bar{x}) \leq 0$ , because otherwise the first  $\varepsilon$ -order quantity  $\|\Delta \hat{x} - \Delta x\|m$  will be favorable to Helen. Then considering  $\Delta \hat{x} \cdot n(\bar{x}) \leq 0$ , we have  $\Delta \hat{x} = \Delta x$  and by symmetry of the quadratic form associated to  $(D^2u - \Gamma)$ , the optimization problem (1.2.31) reduces to

$$\begin{aligned} \max_{\Gamma} \min_{\Delta \hat{x} \cdot n(\bar{x}) \leq 0} & \left[ \frac{1}{2} \langle (D^2u - \Gamma) \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, u, Du, \Gamma) \right] \\ & = \varepsilon^2 \max_{\Gamma} \min_{\Delta \hat{x}} \left[ \frac{1}{2} \varepsilon^{-2} \langle (D^2u - \Gamma) \Delta \hat{x}, \Delta \hat{x} \rangle - \mathcal{F}(t, x, u, Du, \Gamma) \right]. \end{aligned} \quad (1.2.40)$$

Helen should choose  $\Gamma \leq D^2u$ , since otherwise Mark can drive  $\varepsilon^{-2} \langle (D^2u - \Gamma) \Delta \hat{x}, \Delta \hat{x} \rangle$  to  $-\infty$  by a suitable choice of  $\Delta \hat{x}$ . Thus, the min attainable by Mark is zero and is at least realized for the choice  $\Delta \hat{x} = 0$ . Helen's maximization reduces to

$$\max_{\Gamma \leq D^2u} [u_t - \mathcal{F}(t, x, u, Du, \Gamma)].$$

Since the PDE is parabolic, i.e. since  $\mathcal{F}$  satisfies (1.1.4), Helen's optimal choice is  $\Gamma = D^2u$  and (1.2.28) reduces formally to  $u_t - \mathcal{F}(t, x, u, Du, D^2u) = 0$ .

If  $m < 0$ , Helen must now choose  $\Gamma$ . In fact, we are going to see that the choice of  $p_1 = p \cdot n(\bar{x})$  obtained at (1.2.38) can be slightly improved by taking into account the additional terms containing  $D^2u$  and  $\Gamma$ . Suppose Helen chooses  $p$  such that  $(p - Du)|_{V^\perp} = 0$  (notice that our first order computation (1.2.38) fulfills this condition) and Mark chooses a move  $\Delta\hat{x}^*$  realizing the minimum on  $\Delta\hat{x}$  in (1.2.31). We consider two cases depending on  $\Delta\hat{x}^*$ .

**Case a:** if  $\Delta\hat{x}^* \in V^\perp$ , we can restrain the minimization problem to the moves  $\Delta\hat{x}$  which belong to  $V^\perp$ ,  $\Delta\hat{x} = \Delta x$ . Thus, the optimization problem (1.2.31) reduces to computing

$$\mathcal{M}_{V^\perp} = \varepsilon^2 \max_{\Gamma} \min_{\substack{\Delta\hat{x} \in V^\perp \\ \|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}}} \left[ \frac{1}{2} \varepsilon^{-2} \langle (D^2u - \tilde{\Gamma}) \Delta x, \Delta x \rangle - \mathcal{F}(t, x, u, p, \Gamma) \right],$$

where  $\tilde{\Gamma} = \Gamma|_{V^\perp}$ . Helen should choose  $\tilde{\Gamma} \leq \widetilde{D^2u}$ , since otherwise Mark can drive  $\varepsilon^{-2} \langle (D^2u - \tilde{\Gamma}) \Delta x, \Delta x \rangle$  to  $-\infty$  by a suitable choice of  $\Delta\hat{x}$ . By repeating the same argument of ellipticity of  $\mathcal{F}$  already used for  $m > 0$ , Helen's optimal choice is  $\tilde{\Gamma} = \widetilde{D^2u}$ .

**Case b:** if  $\Delta\hat{x}^* \notin V^\perp$ , there exists a unit vector  $v$  orthogonal to  $n(\bar{x})$  such that  $\Delta\hat{x}^* \in \text{span}(n(\bar{x}), v)$ . Thus, we restrain the minimization problem on  $\Delta\hat{x}$  given by (1.2.31) to the moves  $\Delta\hat{x}$  which belong to the disk  $D = \text{span}(n(\bar{x}), v) \cap B(\varepsilon^{1-\alpha})$ . This gives the optimization problem  $\mathcal{M}_D$  given by

$$\mathcal{M}_D = \max_{p_1, \Gamma} \min_{\Delta\hat{x} \in D} \left[ (Du - p)_1 (\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| m + \frac{1}{2} \langle D^2u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, u, p, \Gamma) \right] \quad (1.2.41)$$

by taking into account that  $\tilde{p} = \widetilde{Du}$  and  $\tilde{\Gamma} = \widetilde{D^2u}$ . Neglecting  $-\varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma)$ , we want to compute  $\max_{s_p, \Gamma} \min_{\Delta\hat{x}} \mathcal{N}(s_p, \Gamma, \Delta\hat{x})$  with

$$\mathcal{N}(s_p, \Gamma, \Delta\hat{x}) = s_p (\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| m + \frac{1}{2} \langle D^2u \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle. \quad (1.2.42)$$

Since  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , we parametrize the disk  $D$  by  $\Delta\hat{x} = \lambda \varepsilon^{1-\alpha} n(\bar{x}) + \mu \varepsilon^{1-\alpha} v$  with  $\lambda^2 + \mu^2 \leq 1$ . Notice that the calculation of  $\Gamma_{1v} := \langle \Gamma n(\bar{x}), v \rangle$  for  $v$  orthogonal to  $n(\bar{x})$  implies the computation of  $(\Gamma n(\bar{x}))|_{V^\perp}$ . If Mark chooses  $\Delta\hat{x}$  such that  $\lambda \geq \lambda_0$  for which the boundary is crossed,

$$\begin{aligned} \mathcal{N}(s_p, \Gamma, \Delta\hat{x}) &= (s_p + m) \varepsilon^{1-\alpha} \lambda - \text{dist}(x) m + \frac{1}{2} \text{dist}^2(x) (D^2u)_{11} - \frac{1}{2} \lambda^2 \varepsilon^{2-2\alpha} \Gamma_{11} \\ &\quad - \mu \left( \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} (D^2u)_{1v} - \lambda \Gamma_{1v} \right) \varepsilon^{2-2\alpha}, \end{aligned} \quad (1.2.43)$$

whereas for  $\Delta\hat{x}$  such that  $\lambda \leq \lambda_0$  for which the boundary is not crossed,

$$\mathcal{N}(s_p, \Gamma, \Delta\hat{x}) = s_p \varepsilon^{1-\alpha} \lambda + \frac{1}{2} \lambda^2 ((D^2u)_{11} - \Gamma_{11}) \varepsilon^{2-2\alpha} - \lambda \mu ((D^2u)_{1v} - \Gamma_{1v}) \varepsilon^{2-2\alpha}. \quad (1.2.44)$$

For fixed  $\lambda$ , Mark will always choose  $\mu$  so that the last term is negative and maximal which leads to

$$\mu = \begin{cases} \text{sgn}\left(\frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} (D^2u)_{1v} - \lambda \Gamma_{1v}\right) \sqrt{1 - \lambda^2}, & \text{if } \lambda_0 \leq \lambda \leq 1, \\ \text{sgn}(\lambda ((D^2u)_{1v} - \Gamma_{1v})) \sqrt{1 - \lambda^2}, & \text{if } -1 \leq \lambda < \lambda_0. \end{cases}$$

The min of  $\mathcal{N}(s_p, \Gamma, \Delta\hat{x})$  on  $\mu$  depends only on  $\lambda = \Delta\hat{x} \cdot n(\bar{x})$  and will be denoted below by  $\mathcal{N}(s_p, \Gamma, \lambda)$ . By virtue of (1.2.43) and (1.2.44) it corresponds, for  $\lambda \geq \lambda_0$ , to

$$\begin{aligned} \mathcal{N}(s_p, \Gamma, \lambda) &= (s_p + m) \varepsilon^{1-\alpha} \lambda - \text{dist}(x) m + \frac{1}{2} \text{dist}^2(x) (D^2u)_{11} - \frac{1}{2} \lambda^2 \varepsilon^{2-2\alpha} \Gamma_{11} \\ &\quad - \sqrt{1 - \lambda^2} \left| \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} (D^2u)_{1v} - \lambda \Gamma_{1v} \right| \varepsilon^{2-2\alpha} \end{aligned}$$

and, for  $\lambda \leq \lambda_0$ , to

$$\mathcal{N}(s_p, \Gamma, \lambda) = s_p \varepsilon^{1-\alpha} \lambda + \frac{1}{2} \lambda^2 ((D^2 u)_{11} - \Gamma_{11}) \varepsilon^{2-2\alpha} - |\lambda| \sqrt{1 - \lambda^2} |(D^2 u)_{1v} - \Gamma_{1v}| \varepsilon^{2-2\alpha}.$$

The second order terms containing  $D^2 u$  and  $\Gamma$  being a little perturbation for  $\varepsilon > 0$  small enough compared to  $(\varepsilon^{1-\alpha} - \text{dist}(x))m$  for  $\text{dist}(x) \ll \varepsilon^{1-\alpha}$ , it is sufficient to consider the case (C6) which led to (1.2.38) and  $(\lambda_1, \lambda_2) = (1, -1)$  corresponding to  $\Delta \hat{x} = \pm \varepsilon^{1-\alpha} n(\bar{x})$ . Therefore, we are going to compare the moves close to the optimal choices  $\Delta \hat{x} = \pm \varepsilon^{1-\alpha} n(\bar{x})$  previously obtained by considering only the first terms in the Taylor expansion. More precisely, we may assume  $\lambda \approx \pm 1$ , which leads to making the change of variables  $\lambda_1 = 1 - \rho_1$ ,  $\lambda_2 = -1 + \rho_2$  and take  $\rho_i \xrightarrow{\varepsilon \rightarrow 0} 0$  for  $i = 1, 2$ . After some computations, we get a Taylor expansion in  $\rho_i$ ,  $i = 1, 2$ , in the form

$$\begin{aligned} \mathcal{N}(s_p, \Gamma, 1 - \rho_1) &= (s_p + m) \varepsilon^{1-\alpha} - \text{dist}(x)m + \frac{1}{2} \text{dist}^2(x) (D^2 u)_{11} - \frac{1}{2} \varepsilon^{2-2\alpha} \Gamma_{11} \\ &\quad - \sqrt{2\rho_1} \left| \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} (D^2 u)_{1v} - \Gamma_{1v} \right| \varepsilon^{2-2\alpha} + \rho_1 \varepsilon^{1-\alpha} [-(s_p + m) + \varepsilon^{1-\alpha} \Gamma_{11}] + O(\varepsilon^{2-2\alpha} \rho_1^{3/2}), \end{aligned} \quad (1.2.45)$$

and

$$\begin{aligned} \mathcal{N}(s_p, \Gamma, -1 + \rho_2) &= -s_p \varepsilon^{1-\alpha} - \frac{1}{2} ((D^2 u)_{11} - \Gamma_{11}) \varepsilon^{2-2\alpha} \\ &\quad - \sqrt{2\rho_2} |(D^2 u)_{1v} - \Gamma_{1v}| \varepsilon^{2-2\alpha} + \rho_2 \varepsilon^{1-\alpha} (s_p - ((D^2 u)_{11} - \Gamma_{11}) \varepsilon^{1-\alpha}) + O(\varepsilon^{2-2\alpha} \rho_2^{3/2}). \end{aligned} \quad (1.2.46)$$

First of all, we are now going to focus on the 0-order terms on the  $\rho_1, \rho_2$  variables. Dropping the next terms corresponds to the two particular moves  $\Delta \hat{x} = \pm \varepsilon^{1-\alpha} n(\bar{x})$  ( $\rho_1 = \rho_2 = 0$ ). For fixed  $\Gamma$ , since these terms containing  $\Gamma$  have the same contributions, we can omit the dependence of  $\mathcal{N}(\cdot, \Gamma, 1)$  and  $\mathcal{N}(\cdot, \Gamma, -1)$  on  $\Gamma$ . Then, by repeating the same arguments already used, there exists a unique  $s_2^*$  realizing the max of  $\min(\mathcal{N}(\cdot, \Gamma, 1), \mathcal{N}(\cdot, \Gamma, -1))$  for which both functions are equal. After some calculations, we find

$$s_2^* = -\frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m + \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2 u)_{11}. \quad (1.2.47)$$

If  $m < 0$ , Helen will finally choose

$$p_{\text{opt}}(x) = Du + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m - \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2 u)_{11} \right] n(\bar{x}). \quad (1.2.48)$$

To complete the analysis on the 0-order terms on the  $\rho_1, \rho_2$  variables, we are now going to see how Helen must choose  $\Gamma_{11}$ . By conserving only the 0-order terms, we obtain

$$\mathcal{M}_D \approx \frac{1}{2} (\varepsilon^{1-\alpha} - \text{dist}(x))m + \max_{\Gamma_{11}} \left[ \frac{1}{4} (\varepsilon^{2-2\alpha} + \text{dist}^2(x)) (D^2 u)_{11} - \frac{1}{2} \varepsilon^{2-2\alpha} \Gamma_{11} - \varepsilon^2 \mathcal{F}(t, x, u, p_{\text{opt}}, \Gamma) \right].$$

Since  $\Gamma_{11}$  cannot counterbalance the first order term, the  $\Gamma_{11}$ -term and the second order terms are gathered. Helen wants to make the best choice, so she is going to choose  $\Gamma_{11}$  such that

$$\frac{1}{4} (\varepsilon^{2-2\alpha} + \text{dist}^2(x)) (D^2 u)_{11} - \frac{1}{2} \varepsilon^{2-2\alpha} \Gamma_{11} \geq 0.$$

By ellipticity of  $\mathcal{F}$ , Helen will choose  $\Gamma_{11}$  such that this upper bound on  $\Gamma_{11}$  is attained. She takes

$$\Gamma_{11} = \frac{1}{2} \left( 1 + \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2 u)_{11}. \quad (1.2.49)$$

It remains to determine  $\Gamma_{1v}$ . By plugging the optimal choices  $s_2^*$ , corresponding to  $p_{\text{opt}}$ , and  $\Gamma_{11}$ , respectively given by (1.2.47) and (1.2.49) in (1.2.45)–(1.2.46), we have

$$\begin{aligned} \mathcal{N}(s_2^*, \Gamma, 1 - \rho_1) &= \frac{1}{2} (\varepsilon^{1-\alpha} - \text{dist}(x))m - \sqrt{2\rho_1} \left| \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} (D^2 u)_{1v} - \Gamma_{1v} \right| \varepsilon^{2-2\alpha} \\ &\quad - (s_2^* + m) \rho_1 \varepsilon^{1-\alpha} + O(\varepsilon^{2-2\alpha} \rho_1), \end{aligned}$$

and

$$\mathcal{N}(s_2^*, \Gamma, -1 + \rho_2) = \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m - \sqrt{2\rho_2} |(D^2u)_{1v} - \Gamma_{1v}| \varepsilon^{2-2\alpha} + s_2^* \rho_2 \varepsilon^{1-\alpha} + O(\varepsilon^{2-2\alpha} \rho_2).$$

Dropping the  $O(\varepsilon^{2-2\alpha} \rho_i)$  terms and noticing that  $s_2^* > 0$  and  $-(s_2^* + m) > 0$  for  $\varepsilon$  small enough, the two minimization problems  $\min_{\rho_i} \mathcal{N}(s_2^*, \Gamma, 1 - \rho_i)$ ,  $i \in \{1, 2\}$  for Mark reduce to find

$$\min_{0 < \rho \leq 1} f(\rho), \quad \text{where } f(\rho) = a\sqrt{\rho} + b\rho,$$

with  $a < 0 < b$ . Differentiating  $f$ , the minimum of  $f$  is attained at  $\sqrt{\rho^*} = -\frac{a}{2b}$ . We can notice that this computation is equivalent to formally differentiating the Taylor expansion of (1.2.43)–(1.2.44). Conserving the predominant terms and dropping the next terms, the minimum of  $\mathcal{N}(s_2^*, \Gamma, 1 - \rho_1)$  and  $\mathcal{N}(s_2^*, \Gamma, -1 + \rho_2)$  are respectively attained at

$$\sqrt{\rho_1^*} \simeq \frac{1}{\sqrt{2}} \frac{|\frac{d}{\varepsilon^{1-\alpha}}(D^2u)_{1v} - \Gamma_{1v}|}{|s_2^* + m|} \varepsilon^{1-\alpha} \quad \text{and} \quad \sqrt{\rho_2^*} \simeq \frac{1}{\sqrt{2}} \frac{|(D^2u)_{1v} - \Gamma_{1v}|}{|s_2^*|} \varepsilon^{1-\alpha}.$$

Assuming formally that these approximations are in fact equalities, we obtain

$$\mathcal{N}(s_2^*, \Gamma, 1 - \rho_1^*) = \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m - \frac{1}{2} \left| \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}(D^2u)_{1v} - \Gamma_{1v} \right|^2 \frac{\varepsilon^{3-3\alpha}}{|s_2^* + m|} + O(\varepsilon^{4-4\alpha}), \quad (1.2.50)$$

$$\mathcal{N}(s_2^*, \Gamma, -1 + \rho_2^*) = \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m - \frac{1}{2} |(D^2u)_{1v} - \Gamma_{1v}|^2 \frac{\varepsilon^{3-3\alpha}}{|s_2^*|} + O(\varepsilon^{4-4\alpha}). \quad (1.2.51)$$

Helen now has to choose  $\Gamma_{1v}$  such that  $\min\{\mathcal{N}(s_2^*, \Gamma, 1 - \rho_1^*), \mathcal{N}(s_2^*, \Gamma, -1 + \rho_2^*)\}$  is maximal. We could compute the optimal value of  $\Gamma_{1v}$  on the  $\varepsilon^{3-3\alpha}$ -terms. However, it is not very useful. Since  $m$  is a constant and  $\varepsilon^{3-3\alpha} \ll \varepsilon^2$  by (1.2.13), the  $\varepsilon^{3-3\alpha}$ -terms are negligible compared to  $-\varepsilon^2 f(t, x, u, p_{\text{opt}}, \Gamma)$  that we have omitted until now. For instance Helen can fix  $\Gamma_{1v}$  such that one of the two terms depending on  $\Gamma_{1v}$  in (1.2.50) and (1.2.51) is equal to zero:  $\Gamma_{1v} = (D^2u)_{1v}$  or  $\Gamma_{1v} = \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}(D^2u)_{1v}$ . The two choices are equivalent because Mark can reverse his move  $\Delta\hat{x}$ . For sake of simplicity, we assume Helen chooses  $\Gamma_{1v} = (D^2u)_{1v}$ . It is worth noticing that this expansion holds if  $m$  is far from zero and we shall modify our arguments very carefully in Section 1.4 when  $m$  is negative but small with respect to a certain power of  $\varepsilon$ .

Thus, if  $m < 0$ , Helen will choose

$$\Gamma_{\text{opt}}(x) = D^2u + \left[ \frac{1}{2} \left( -1 + \frac{d^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2u)_{11} \right] E_{11}. \quad (1.2.52)$$

Unlike the usual game [88], when Helen chooses  $p$  and  $\Gamma$  optimally, she does not become indifferent to Mark's choice of  $\Delta\hat{x}$ . More precisely, it depends on the projection of  $\Delta\hat{x}$  with respect to  $n(\bar{x})$ . Our games always have this feature.

**Step 3:** Now let us go back to the original optimization problem (1.2.28). If  $m = 0$ , by letting  $\varepsilon \rightarrow 0$ , we get  $h(x) - Du(x) \cdot n(x) = 0$ . Otherwise, (1.2.28) formally reduces to

$$0 \approx \varepsilon^2 u_t + \begin{cases} \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m - \varepsilon^2 \mathcal{F}(t, x, u, p_{\text{opt}}(x), \Gamma_{\text{opt}}(x)) + o(\varepsilon^2), & \text{if } \text{dist}(x) \leq \varepsilon^{1-\alpha} \text{ and } m < 0, \\ -\varepsilon^2 \mathcal{F}(t, x, u, Du, D^2u), & \text{if } \text{dist}(x) \geq \varepsilon^{1-\alpha} \text{ or } m > 0, \end{cases} \quad (1.2.53)$$

with  $p_{\text{opt}}$  and  $\Gamma_{\text{opt}}$  respectively defined by (1.2.48) and (1.2.52). If  $x \in \Omega$ , for  $\varepsilon$  small enough, the second relation in (1.2.53) is always valid so that we deduce from the  $\varepsilon^2$ -order terms in (1.2.53) that  $u_t - \mathcal{F}(t, x, u, Du, D^2u) = 0$ . If  $x \in \partial\Omega$ ,  $\text{dist}(x) = 0$  and we distinguish the cases  $m > 0$  and  $m < 0$ . If  $m > 0$ , one more time the second relation in (1.2.53) is always valid so that  $u_t - \mathcal{F}(t, x, u, Du, D^2u) = 0$ . Otherwise, if  $m < 0$ , the first relation in (1.2.53) is always satisfied. We observe that the  $\varepsilon$ -order term is predominant since  $\varepsilon^{1-\alpha} \gg \varepsilon^2$ . By dividing by  $\varepsilon^{1-\alpha}$  and letting  $\varepsilon \rightarrow 0$ , we obtain  $m = 0$  that leads to a contradiction since we assumed  $m < 0$ . Therefore, we have formally shown that on the boundary  $h(x) - Du(x) \cdot n(x) = 0$  or  $u_t - \mathcal{F}(t, x, u, Du, D^2u) = 0$ .

### 1.2.2.2 Main parabolic result

We shall show, under suitable hypotheses, that  $\bar{u}$  and  $\underline{v}$  are respectively viscosity sub and supersolutions. A natural question is to compare  $\bar{u}$  and  $\underline{v}$ . This is a global question, which we can answer only if the PDE has a comparison principle. Such a principle asserts that if  $u$  is a subsolution and  $v$  is a supersolution then  $u \leq v$ . If the PDE has such a principle then it follows that  $\bar{u} \leq \underline{v}$ . The opposite inequality is immediate from the definitions, so it follows that  $\bar{u} = \underline{v}$ , and we get a viscosity solution of the PDE. It is in fact the unique viscosity solution, since the comparison principle implies uniqueness.

**Theorem 1.2.4.** *Consider the final-value problem (1.2.12) where  $\mathcal{F}$  satisfies (1.1.4)–(1.1.7),  $g$  and  $h$  are continuous, uniformly bounded, and  $\Omega$  is a  $C^2$ -domain satisfying both the uniform interior and exterior ball conditions. Assume the parameters  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15). Then  $\bar{u}$  and  $\underline{v}$  are uniformly bounded on  $\bar{\Omega} \times [t_*, T]$  for any  $t_* < T$ , and they are respectively a viscosity subsolution and a viscosity supersolution of (1.2.12). If the PDE has a comparison principle (for uniformly bounded solutions), then it follows that  $u^\varepsilon$  and  $v^\varepsilon$  converge locally uniformly to the unique viscosity solution of (1.2.12).*

This theorem is an immediate consequence of Propositions 1.3.4 and 1.5.1.

In this theorem, we require the domain  $\Omega$  to be  $C^2$ . This assumption is crucial for the proof of Proposition 1.3.4 case (iii) corresponding to the convergence at the final time in the viscosity sense (see Remark 1.3.5). It can also be noticed that it is this part of Proposition 1.3.4 which allows to use a comparison principle for the parabolic PDE. On the other hand, since the game already requires the uniform interior and exterior ball conditions, the domain  $\Omega$  is in fact at least  $C^{1,1}$ . It remains an open question to overcome the analysis in this case.

As mentioned in [88], some sufficient conditions for the PDE to have a comparison result can be found in Section 4.3 of [42]. In our framework, we can emphasise on the comparison principle obtained by Sato [125, Theorem 2.1] for a fully nonlinear parabolic equation with a homogeneous condition. The reader is also referred to the introduction for other references about comparison and existence results. Note that most comparison results require  $f(t, x, z, p, \Gamma)$  to be nondecreasing in  $z$ .

We close this section with the observation that if  $U^\varepsilon(x, z, t)$  is a strictly decreasing function of  $z$  then  $v^\varepsilon(x, t) = u^\varepsilon(x, t)$ . A sufficient condition for this to hold is that  $\mathcal{F}$  be nondecreasing in  $z$ :

**Lemma 1.2.5.** *Suppose  $\mathcal{F}$  is non-decreasing in  $z$  in the sense that*

$$\mathcal{F}(t, x, z_1, p, \Gamma) \geq \mathcal{F}(t, x, z_0, p, \Gamma) \quad \text{whenever } z_1 > z_0.$$

*Then  $U^\varepsilon$  satisfies*

$$U^\varepsilon(x, z_1, t_j) \leq U^\varepsilon(x, z_0, t_j) - (z_1 - z_0) \quad \text{whenever } z_1 > z_0,$$

*at each discrete time  $t_j = T - j\varepsilon^2$ . In particular,  $U^\varepsilon$  is strictly decreasing in  $z$  and  $v^\varepsilon = u^\varepsilon$ .*

*Proof.* The whole space case is provided in [88, Lemma 2.4]. For our game, it suffices to add  $-\|\Delta\hat{x} - \Delta x\|h(x + \Delta x)$  in the expressions of  $\Delta z_0$  and  $\Delta z_1$  defined in the proof of [88, Lemma 2.4]. The rest of the proof remains unchanged.  $\square$

### 1.2.3 Nonlinear elliptic equations

This section explains how our game can be used to solve stationary problems with Neumann boundary conditions. The framework is similar to the parabolic case, but one new issue arises: we must introduce discounting as in [88], to be sure Helen's value function is finite. Therefore we focus on

$$\begin{cases} \mathcal{F}(x, u, Du, D^2u) + \lambda u = 0, & \text{in } \Omega, \\ \langle Du, n \rangle = h, & \text{on } \partial\Omega, \end{cases} \quad (1.2.54)$$

where  $\Omega$  is a domain with  $C^2$ -boundary and satisfies both the uniform interior and exterior ball condition presented in the introduction. The constant  $\lambda$  (which plays the role of an interest rate) must be positive, and large enough so that (1.1.8) holds. Notice that if  $\mathcal{F}$  is independent of  $z$  then any  $\lambda$  will do.

We now present the game. The main difference with Section 1.2.2 is the presence of discounting. The boundary condition  $h$  is assumed to be a bounded continuous function on  $\partial\Omega$ . Besides the parameters  $\alpha, \beta, \gamma$  introduced previously, in the stationary case we need two new parameters,  $m$  and  $M$ , and a  $C_b^2(\overline{\Omega})$ -function  $\psi$  such that

$$\frac{\partial\psi}{\partial n} = \|h\|_\infty + 1 \quad \text{on } \partial\Omega. \quad (1.2.55)$$

It suffices to construct  $\psi_1$  such that it is  $C_b^2(\overline{\Omega})$  and satisfies  $\frac{\partial\psi_1}{\partial n} = 1$  on the boundary. Then we can define  $\psi$  by  $\psi = (\|h\|_\infty + 1)\psi_1$ . The existence and construction of such a function  $\psi_1$  for a  $C^2$ -domain  $\Omega$  satisfying the uniform interior ball condition is discussed at the end of this section.

From  $m$  and  $\psi$  we construct a function  $\chi$  defined by

$$\chi(x) = m + \|\psi\|_{L^\infty(\overline{\Omega})} + \psi(x). \quad (1.2.56)$$

Both  $m$  and  $M$  are positive constants, which also yield that  $\chi$  is positive.  $M$  serves to cap the score, and the function  $\chi$  determines what happens when the cap is reached. We shall in due course choose  $m$  such that  $m + 2\|\psi\|_{L^\infty} = M - 1$  and require that  $M$  is sufficiently large. Like the choices of  $\alpha, \beta, \gamma$ , the parameters  $M, m$  and the function  $\psi$  are used to define the game but they do not influence the resulting PDE. As in Section 1.2.2, we proceed in two steps:

- First we introduce  $U^\varepsilon(x, z)$ , the optimal worst-case present value of Helen's wealth if the initial stock price is  $x$  and her initial wealth is  $-z$ .
- Then we define  $u^\varepsilon(x)$  and  $v^\varepsilon(x)$  as the maximal and minimal initial debt Helen should have at time  $t$  to break even upon exit.

The definition of  $U^\varepsilon(x, z)$  for  $x \in \overline{\Omega}$  involves a game similar to that of the last section:

1. Initially, at time  $t_0 = 0$ , the stock price is  $x_0 = x$  and Helen's debt is  $z_0 = z$ .
2. Suppose, at time  $t_j = j\varepsilon^2$ , the stock price is  $x_j$  and Helen's debt is  $z_j$  with  $|z_j| < M$ . Then Helen chooses a vector  $p_j \in \mathbb{R}^d$  and a matrix  $\Gamma_j \in \mathcal{S}_d$ , restricted in magnitude by (1.2.16). Knowing these choices, Mark determines the next stock price  $x_{j+1} = x_j + \Delta x$  so as to degrade Helen's outcome. The increment  $\Delta x$  allows to model the reflection exactly as in the previous subsections. Mark chooses an intermediate point  $\hat{x}_{j+1} = x_j + \Delta \hat{x}_j \in \mathbb{R}^d$  such that

$$\|\Delta \hat{x}_j\| \leq \varepsilon^{1-\alpha}.$$

This position  $\hat{x}_{j+1}$  determines the new position  $x_{j+1} = x_j + \Delta x_j$  at time  $t_{j+1}$  by

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}).$$

Helen experiences a loss at time  $t_j$  of

$$\delta_j = p_j \cdot \Delta \hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta \hat{x}_j, \Delta \hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(x_j, z_j, p_j, \Gamma_j) - \|\Delta \hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j). \quad (1.2.57)$$

As a consequence, her time  $t_{j+1} = t_j + \varepsilon^2$  debt becomes

$$z_{j+1} = e^{\lambda \varepsilon^2} (z_j + \delta_j),$$

where the factor  $e^{\lambda \varepsilon^2}$  takes into account her interest payments.

3. If  $z_{j+1} \geq M$ , then the game terminates, and Helen pays a “termination-by-large-debt penalty” worth  $e^{\lambda \varepsilon^2} (\chi(x_j) - \delta_j)$  at time  $t_{j+1}$ . Similarly, if  $z_{j+1} \leq -M$ , then the game terminates, and Helen receives a “termination-by-large-wealth bonus” worth  $e^{\lambda \varepsilon^2} (\chi(x_j) + \delta_j)$  at time  $t_{j+1}$ . If the game stops this way, we call  $t_{j+1}$  the “ending index”  $t_K$ .



4. If the game has not terminated then Helen and Mark repeat this procedure at time  $t_{j+1} = t_j + \varepsilon^2$ .  
 If the game never stops, the “ending index”  $t_K$  is  $+\infty$ .

Helen’s goal is a bit different from before, due to the presence of discounting: she seeks to maximize the minimum present value of her future income, using the discount factor of  $e^{-j\lambda\varepsilon^2}$  for income received at time  $t_j$ . If the game ends by capping at time  $t_K$  with  $z_K \geq M$ , then the present value of her income is

$$\begin{aligned} U^\varepsilon(x_0, z_0) &= -z_0 - \delta_0 - e^{-\lambda\varepsilon^2}\delta_1 - \dots - e^{-(K-1)\lambda\varepsilon^2}\delta_{K-1} - e^{-(K-1)\lambda\varepsilon^2}(\chi(x_{K-1}) - \delta_{K-1}) \\ &= e^{-(K-1)\lambda\varepsilon^2}(-z_{K-1} - \chi(x_{K-1})). \end{aligned}$$

Similarly, if the game ends by capping at time  $t_K$  with  $z_K \leq -M$ , then the present value of her income is

$$\begin{aligned} U^\varepsilon(x_0, z_0) &= -z_0 - \delta_0 - e^{-\lambda\varepsilon^2}\delta_1 - \dots - e^{-(K-1)\lambda\varepsilon^2}\delta_{K-1} + e^{-(K-1)\lambda\varepsilon^2}(\chi(x_{K-1}) + \delta_{K-1}) \\ &= e^{-(K-1)\lambda\varepsilon^2}(-z_{K-1} + \chi(x_{K-1})). \end{aligned}$$

If the game never ends (since  $z_j$  and  $\chi(x_j)$  are uniformly bounded), we can take  $K = \infty$  in the preceding formula to see that the present value of her income is 0.

To get a dynamic programming characterization of  $U^\varepsilon$ , we observe that if  $|z_0| < M$  then

$$U^\varepsilon(x_0, z_0) = \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda\varepsilon^2} U^\varepsilon(x_1, z_1), & \text{if } |z_1| < M, \\ -z_0 - \chi(x_0), & \text{if } z_1 \geq M, \\ -z_0 + \chi(x_0), & \text{if } z_1 \leq -M. \end{cases}$$

Since the game is stationary (nothing distinguishes time 0), the associated dynamic programming principle is that for  $|z| < M$ ,

$$U^\varepsilon(x, z) = \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda\varepsilon^2} U^\varepsilon(x', z'), & \text{if } |z'| < M, \\ -z - \chi(x), & \text{if } z' \geq M, \\ -z + \chi(x), & \text{if } z' \leq -M, \end{cases} \quad (1.2.58)$$

where  $x' = \text{proj}_{\overline{\Omega}}(x + \Delta \hat{x})$  and  $z' = e^{\lambda\varepsilon^2}(z + \delta)$ , with  $\delta$  defined as in (3.3.1). Here  $p$ ,  $\Gamma$  and  $\Delta \hat{x}$  are constrained as usual by (1.2.16)–(1.2.17), and we write sup/inf rather than max/min since it is no longer clear that the optima are achieved (since the right-hand side is now a discontinuous function of  $p$ ,  $\Gamma$  and  $\Delta \hat{x}$ ). The preceding discussion defines  $U^\varepsilon$  only for  $|z| < M$ ; it is natural to extend the definition to all  $z$  by

$$U^\varepsilon(x, z) = \begin{cases} -z - \chi(x), & \text{for } z \geq M, \\ -z + \chi(x), & \text{for } z \leq -M, \end{cases}$$

which corresponds to play being “capped” immediately. Notice that when extended this way,  $U^\varepsilon$  is strictly negative for  $z \geq M$  and strictly positive for  $z \leq -M$ .

The definitions of  $u^\varepsilon$  and  $v^\varepsilon$  are slightly different from those in Section 1.2.2:

$$u^\varepsilon(x_0) = \sup\{z_0 : U^\varepsilon(x_0, z_0) > 0\}, \quad (1.2.59)$$

$$v^\varepsilon(x_0) = \inf\{z_0 : U^\varepsilon(x_0, z_0) < 0\}. \quad (1.2.60)$$

The change from Section 1.2.2 is that the inequalities in (1.2.22)–(1.2.23) are strict.

**Proposition 1.2.6.** *Let  $m_1, M$  be two constants such that  $0 < m_1 < M$ . Then whenever  $x \in \overline{\Omega}$  and  $-m_1 \leq u^\varepsilon(x) < M$  we have*

$$\begin{aligned} u^\varepsilon(x) &\leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ e^{-\lambda\varepsilon^2} u^\varepsilon(x + \Delta x) \right. \\ &\quad \left. - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, u^\varepsilon(x), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right], \quad (1.2.61) \end{aligned}$$

for  $\varepsilon$  small enough (depending on  $m_1$  and the parameters of the game but not on  $x$ ). Similarly, if  $x \in \overline{\Omega}$  and  $-M < v^\varepsilon(x) < m_1$  then for  $\varepsilon$  small enough

$$v^\varepsilon(x) \geq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ e^{-\lambda \varepsilon^2} v^\varepsilon(x + \Delta x) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, v^\varepsilon(x), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right]. \quad (1.2.62)$$

As usual, the sup and inf are constrained by (1.2.16) and (1.2.17) and  $\Delta x$  is determined by (1.2.21).

*Proof.* We shall focus on (1.2.61); the proof for (1.2.62) follows exactly the same lines. Since  $-m_1 \leq u^\varepsilon(x) < M$ , there is a sequence  $z^k \uparrow u^\varepsilon(x)$  such that  $U^\varepsilon(x, z^k) > 0$ . Since  $u^\varepsilon(x)$  is bounded away from  $-M$ , we may suppose that  $z^k$  also remains bounded away from  $-M$ . Dropping the index  $k$  for simplicity of notation, consider any such  $z = z^k$ . The fact that  $U^\varepsilon(x, z) > 0$  tells us that the right-hand side of the dynamic programming principle (3.3.2) is positive. So there exist  $p, \Gamma$  constrained by (1.2.16) such that for any  $\Delta \hat{x}$  satisfying (1.2.17),

$$0 < \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda \varepsilon^2} U^\varepsilon(x', z'), & \text{if } |z'| < M, \\ -z - \chi(x), & \text{if } z' \geq M, \\ -z + \chi(x), & \text{if } z' \leq -M, \end{cases} \quad (1.2.63)$$

where  $x' = \text{proj}_{\overline{\Omega}}(x + \Delta \hat{x})$  and  $z' = e^{\lambda \varepsilon^2}(z + \delta)$ . Capping above, the alternative  $z' \geq M$ , cannot happen, since otherwise we compute

$$-z - \chi(x) = -e^{-\lambda \varepsilon^2} z' - \delta - \chi(x) \leq -M e^{-\lambda \varepsilon^2} - \delta - m \leq -\delta - m < 0,$$

for  $\varepsilon$  small enough because  $\delta$  is bounded by a positive power of  $\varepsilon$ . This sign is a contradiction to our assumption (1.2.63). If  $\varepsilon$  is sufficiently small, capping below (the alternative  $z' \leq -M$ ) cannot occur either, because  $z$  is bounded away from  $-M$  and  $\delta$  is bounded by a positive power of  $\varepsilon$ . Therefore only the first case can take place

$$0 < U^\varepsilon(x + \Delta x, e^{\lambda \varepsilon^2}(z + \delta)),$$

whence by the definition of  $u^\varepsilon$  given by (3.3.3), we deduce that

$$u^\varepsilon(x + \Delta x) \geq e^{\lambda \varepsilon^2}(z + \delta).$$

Thus, we have shown the existence of  $p, \Gamma$  such that for every  $\Delta \hat{x}$ ,

$$z \leq e^{-\lambda \varepsilon^2} u^\varepsilon(x + \Delta x) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right). \quad (1.2.64)$$

Recalling that  $z = z^k \uparrow u^\varepsilon(x)$ , we pass to the limit on both sides of (1.2.64), with  $p, \Gamma$  held fixed, to see that

$$u^\varepsilon(x) \leq e^{-\lambda \varepsilon^2} u^\varepsilon(x + \Delta x) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, u^\varepsilon(x), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right).$$

Since this is true for some  $p, \Gamma$  and for every  $\Delta \hat{x}$ , we have established (1.2.61).  $\square$

The PDE (1.2.54) is the formal Hamilton-Jacobi-Bellman equation associated with the dynamic programming inequalities (1.2.61)–(1.2.62), by the usual Taylor expansion, if one accepts  $-M < v^\varepsilon \approx u^\varepsilon < M$ . Rather than giving that heuristic argument which is quite similar to the one proposed in the parabolic setting, we now state our main result in the stationary setting, which follows from the results in Sections 1.4 and 1.5. It concerns the upper and lower relaxed semi-limits, defined for any  $x \in \overline{\Omega}$ , by

$$\overline{u}(x) = \limsup_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} u^\varepsilon(y) \quad \text{and} \quad \underline{v}(x) = \liminf_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} v^\varepsilon(y), \quad (1.2.65)$$

with the convention that  $y$  approaches  $x$  from  $\overline{\Omega}$  (since  $u^\varepsilon$  and  $v^\varepsilon$  are defined on  $\overline{\Omega}$ ).

**Theorem 1.2.7.** *Consider the stationary boundary value problem (1.2.54) where  $\mathcal{F}$  satisfies (1.1.4) and (1.1.8)–(1.1.10),  $g$  and  $h$  are continuous, uniformly bounded, and  $\Omega$  is a  $C^2$ -domain satisfying both the uniform interior and exterior ball conditions. Assume the parameters of the game  $\alpha, \beta, \gamma$  fulfill (1.2.13)–(1.2.15),  $\psi \in C^2(\overline{\Omega})$  satisfies (1.2.55),  $\chi \in C^2(\overline{\Omega})$  is defined by (1.2.56),  $M$  is sufficiently large and  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ . Then  $u^\varepsilon$  and  $v^\varepsilon$  are well-defined when  $\varepsilon$  is sufficiently small, and they satisfy  $|u^\varepsilon| \leq \chi$  and  $|v^\varepsilon| \leq \chi$ . Their relaxed semi-limits  $\underline{u}$  and  $\underline{v}$  are respectively a viscosity subsolution and a viscosity supersolution of (1.2.54). If in addition we have  $\underline{v} \leq \underline{u}$  and the PDE has a comparison principle, then it follows that  $u^\varepsilon$  and  $v^\varepsilon$  converge locally uniformly in  $\overline{\Omega}$  to the unique viscosity solution of (1.2.54).*

This is an immediate consequence of Propositions 1.3.6 and 1.5.5. A sufficient condition for  $\underline{v} \leq \underline{u}$  is that  $\mathcal{F}$  is nondecreasing in  $z$ . As mentioned in [88], sufficient conditions for the PDE to have a comparison principle can be found for example in Section 5 of [45], and (for more results) in [13]–[19].

Let us now go back to the existence and the construction of  $\psi_1 \in C_b^2(\overline{\Omega})$  such that  $\frac{\partial \psi_1}{\partial n} = 1$  on  $\partial\Omega$ , that we will need at various points of the paper. If  $\Omega$  is of class  $C^2$  and satisfies the uniform interior ball condition of Definition 1.1.2 for a certain  $r$ ,  $d$  is  $C^2$  on  $\Omega(3r/4)$  and an explicit suitable function is

$$\psi_1(x) = \begin{cases} \exp\left[-\frac{\text{dist}(x)}{1-\frac{\text{dist}(x)}{r/2}}\right], & \text{if } \text{dist}(x) < r/2, \\ 0, & \text{if } \text{dist}(x) \geq r/2. \end{cases} \quad (1.2.66)$$

It is clear that  $\text{supp } \psi_1 \subset \Omega(r/2)$ ,  $\psi_1(\overline{\Omega}) \subset [0, 1]$  and  $\psi_1$  is  $C^2$  on  $\Omega(r/2)$ . Then, for all  $x$  such that  $\text{dist}(x) = \frac{r}{2}$ ,  $D\psi_1$  and  $D^2\psi_1$  are continuous at  $x$ . Thus  $\psi_1$  is  $C^2$  on  $\overline{\Omega}$ . It is easy to check that the two first derivatives of  $\psi_1$  are also bounded and that  $\frac{\partial \psi_1}{\partial n} = 1$  on the boundary. Hence, the function  $\psi_1$  defined by (1.2.66) has all the desired properties.

**Remark 1.2.8.** *If  $\Omega$  is a domain with  $C^{2,\alpha}$ -boundary where  $\alpha > 0$ , the Schauder theory [74, Theorem 6.31] ensures the solution  $\psi$  of the elliptic problem*

$$\begin{cases} \Delta\psi - \psi = 0, & \text{in } \Omega, \\ \frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1, & \text{on } \partial\Omega, \end{cases}$$

*is  $C^{2,\alpha}(\overline{\Omega})$ . In addition, the estimate  $\|\psi\|_{C^{2,\alpha}(\overline{\Omega})} \leq C_\Omega(1 + \|h\|_{L^\infty})$  holds for a certain constant  $C_\Omega$  depending only on the domain.*

## 1.3 Convergence

This section presents our main convergence results, linking the limiting behavior of  $v^\varepsilon$  and  $u^\varepsilon$  as  $\varepsilon \rightarrow 0$  to the PDE. The argument uses the framework of [20] and is basically a special case of the theorem proved there.

Convergence is a local issue: in the time-dependent setting, Proposition 1.3.4 shows that in any region where the lower and upper semi-relaxed limits  $\underline{v}$  and  $\underline{u}$  are finite they are in fact viscosity super and subsolutions respectively. The analogous statement for the stationary case is more subtle. In fact, we will need global hypotheses on  $\mathcal{F}$  at Section 1.5.2 to ensure that  $u^\varepsilon$  and  $v^\varepsilon$  are well-defined and satisfy the dynamic programming inequalities (1.2.61)–(1.2.62). Thus, we cannot discuss about  $\underline{v}$  or  $\underline{u}$  without global assumptions on  $\mathcal{F}$ .

### 1.3.1 Viscosity solutions with Neumann condition

Our PDEs can be degenerate parabolic, degenerate elliptic, or even first order equations. Therefore, we cannot expect a classical solution, and we cannot always impose boundary data in the classical sense.

The theory of viscosity solutions provides the proper framework for handling these issues. We review the basic definitions in our setting for the reader's convenience. We refer to [11], [45] and [72] for further details about the general theory. Consider first the final-value problem (1.2.12) in  $\Omega$ ,

$$\begin{cases} -u_t + \mathcal{F}(t, x, u, Du, D^2u) = 0, & \text{for } x \in \Omega \text{ and } t < T, \\ \langle Du(x, t), n(x) \rangle = h(x), & \text{for } x \in \partial\Omega \text{ and } t < T, \\ u(x, T) = g(x), & \text{for } x \in \overline{\Omega}. \end{cases}$$

where  $\mathcal{F}(t, x, z, p, \Gamma)$  is continuous in all its variables and satisfies the monotonicity condition (1.1.4) in its last variable. We must be careful to impose the boundary condition in the viscosity sense.

**Definition 1.3.1.** A real-valued lower-semicontinuous function  $u(x, t)$  defined for  $x \in \overline{\Omega}$  and  $t_* \leq t \leq T$  is a viscosity supersolution of the final-value problem (1.2.12) if

(P1) for any  $(x_0, t_0)$  with  $x_0 \in \Omega$  and  $t_* \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local minimum at  $(x_0, t_0)$ , we have

$$-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0,$$

(P2) for any  $(x_0, t_0)$  with  $x_0 \in \partial\Omega$  and  $t_* \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local minimum on  $\overline{\Omega}$  at  $(x_0, t_0)$ , we have

$$\max\{-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)), \langle D\phi(x_0, t_0), n(x_0) \rangle - h(x_0)\} \geq 0,$$

(P3)  $u \geq g$  at the final time  $t = T$ .

Similarly, a real-valued upper-semicontinuous function  $u(x, t)$  defined for  $x \in \overline{\Omega}$  and  $t_* \leq t \leq T$  is a viscosity subsolution of the final-value problem (1.2.12) if

(P1) for any  $(x_0, t_0)$  with  $x_0 \in \Omega$  and  $t_* \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local maximum at  $(x_0, t_0)$ , we have

$$-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \leq 0,$$

(P2) for any  $(x_0, t_0)$  with  $x_0 \in \partial\Omega$  and  $t_* \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local maximum on  $\overline{\Omega}$  at  $(x_0, t_0)$ , we have

$$\min\{-\partial_t \phi(x_0, t_0) + \mathcal{F}(t_0, x_0, u(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)), \langle D\phi(x_0, t_0), n(x_0) \rangle - h(x_0)\} \leq 0,$$

(P3)  $u \leq g$  at the final time  $t = T$ .

A viscosity solution of (1.2.12) is a continuous function  $u$  that is both a viscosity subsolution and a viscosity supersolution of (1.2.12).

In the stationary problem (1.2.54), the definitions are similar to the time-dependent setting.

**Definition 1.3.2.** A real-valued lower-semicontinuous function  $u(x)$  defined on  $\overline{\Omega}$  is a viscosity supersolution of the stationary problem (1.2.54) if

(E1) for any  $x_0 \in \Omega$  and any smooth  $\phi(x)$  such that  $u - \phi$  has a local minimum at  $x_0$ , we have

$$\mathcal{F}(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) + \lambda u(x_0) \geq 0,$$

(E2) for any  $x_0 \in \partial\Omega$  and any smooth  $\phi(x)$  such that  $u - \phi$  has a local minimum on  $\overline{\Omega}$  at  $x_0$ , we have

$$\max\{\mathcal{F}(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) + \lambda u(x_0), \langle D\phi(x_0), n(x_0) \rangle - h(x_0)\} \geq 0.$$

Similarly, a real-valued upper-semicontinuous function  $u(x)$  defined on  $\overline{\Omega}$  is a viscosity subsolution of the stationary problem (1.2.54) if

(E1) for any  $x_0 \in \Omega$  and any smooth  $\phi(x)$  such that  $u - \phi$  has a local maximum at  $x_0$ , we have

$$\mathcal{F}(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) + \lambda u(x_0) \leq 0,$$

(E2) for any  $x_0 \in \partial\Omega$  and any smooth  $\phi(x)$  such that  $u - \phi$  has a local maximum on  $\overline{\Omega}$  at  $x_0$ , we have

$$\min\{\mathcal{F}(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) + \lambda u(x_0), \langle D\phi(x_0), n(x_0) \rangle - h(x_0)\} \leq 0.$$

A viscosity solution of (1.2.54) is a continuous function  $u$  that is both a viscosity subsolution and a viscosity supersolution of (1.2.54).

In stating these definitions, we have assumed that the final-time data  $g$  and the boundary Neumann condition  $h$  are continuous. In Definition 1.3.1, the pointwise inequality in part (P3) can be replaced by an apparently weaker condition analogous to part (P2). The equivalence of such a definition with the one stated above is standard, the argument uses barriers of the form  $\phi(x, t) = \|x - x_0\|^2/\delta + (T - t)/\mu + K \text{dist}(x)$  with  $\delta$  and  $\mu$  sufficiently small, and is contained in our proof of Proposition 1.3.4 (iii). We shall be focusing on the lower and upper semi-relaxed limits of  $v^\varepsilon$  and  $u^\varepsilon$ , defined by (1.2.26) in the time-dependent setting and (1.2.65) in the stationary case.

We now provide a key definition to deal with the Neumann boundary condition within viscosity solutions framework which will be essential all along the article. We introduce some applications which give bounds on the Neumann penalization term for a smooth function and  $x$  close to the boundary. This approach is well-suited to the viscosity solutions framework. More precisely, we define the applications  $m_\varepsilon$  and  $M_\varepsilon$ , for all  $x \in \Omega(\varepsilon^{1-\alpha})$  and  $\phi \in C^1(\overline{\Omega})$ , by

$$m_\varepsilon^x[\phi] := \inf_{\substack{x+\Delta\hat{x} \notin \Omega \\ \Delta\hat{x}}} \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\}, \quad (1.3.1)$$

$$M_\varepsilon^x[\phi] := \sup_{\substack{x+\Delta\hat{x} \notin \Omega \\ \Delta\hat{x}}} \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\}, \quad (1.3.2)$$

where  $\Delta\hat{x}$  is constrained by (1.2.17) and determines  $\Delta x$  by (1.2.21). Notice that the functions  $m_\varepsilon[\phi]$  and  $M_\varepsilon[\phi]$  are bounded by  $\|h\|_{L^\infty} + \|D\phi\|_{L^\infty}(\overline{\Omega})$ . Since  $h$  is supposed to be continuous, the following property clearly holds.

**Lemma 1.3.3.** *Let  $x \in \partial\Omega$  and  $\phi \in C^1(\overline{\Omega})$ . Suppose there exists a sequence  $(\varepsilon_k, x_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}_+^* \times \overline{\Omega}$  convergent to  $(0, x)$  such that for all  $k$  large enough,  $x_k \in \Omega(\varepsilon_k^{1-\alpha})$ . Then*

$$\lim_{k \rightarrow +\infty} m_{\varepsilon_k}^{x_k}[\phi] = \lim_{k \rightarrow +\infty} M_{\varepsilon_k}^{x_k}[\phi] = h(x) - D\phi(x) \cdot n(x).$$

Similarly, let  $\phi \in C^1(\overline{\Omega} \times [0, T])$ . Suppose there exists a sequence  $(\varepsilon_k, x_k, t_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}_+^* \times \overline{\Omega} \times [0, T]$  convergent to  $(0, x, t)$  such that for all  $k$  large enough,  $x_k \in \Omega(\varepsilon_k^{1-\alpha})$ . Then

$$\lim_{k \rightarrow +\infty} m_{\varepsilon_k}^{x_k}[\phi(\cdot, t_k)] = \lim_{k \rightarrow +\infty} M_{\varepsilon_k}^{x_k}[\phi(\cdot, t_k)] = h(x) - D\phi(x, t) \cdot n(x).$$

### 1.3.2 The parabolic case

We are ready to state our main convergence result in the time-dependent setting. At first sight, the proof seems to use the monotonicity condition (1.1.4). The proof relies on the consistency of the numerical scheme, Propositions 1.4.5, 1.4.10 and 1.4.16, which are proved in Section 1.4. Proposition 1.4.16 is necessary to deal with the degeneration of the consistency estimates due to the Neumann boundary condition. So we also require that  $\mathcal{F}(t, x, z, p, \Gamma)$  satisfy (1.1.6)–(1.1.7), and that the parameters  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15).

**Proposition 1.3.4.** *Suppose  $\mathcal{F}$  and  $\alpha, \beta, \gamma$  satisfy the hypotheses just listed. Assume furthermore that  $\overline{u}$  and  $\underline{v}$  are finite for all  $x$  near  $x_0$  and all  $t \leq T$  near  $t_0$ . Then*

- i. if  $t_0 < T$  and  $x_0 \in \Omega$ , then  $\bar{u}$  is a viscosity subsolution at  $(x_0, t_0)$  and  $\underline{v}$  is a supersolution at  $(x_0, t_0)$  (i.e. each one satisfies part (P1) of the relevant half of Definition 1.3.1 at  $(x_0, t_0)$ ).
- ii. if  $t_0 < T$  and  $x_0 \in \partial\Omega$ , then  $\bar{u}$  is a viscosity subsolution at  $(x_0, t_0)$  and  $\underline{v}$  is a supersolution at  $(x_0, t_0)$  (i.e. each one satisfies part (P2) of the relevant half of Definition 1.3.1 at  $(x_0, t_0)$ ).
- iii. if  $t_0 = T$ , then  $\bar{u}(x_0, T) = g(x_0)$  and  $\underline{v}(x_0, T) = g(x_0)$  (in particular, each one satisfies part (P3) of the relevant half of Definition 1.3.1 at  $(x_0, t_0)$ ).

In particular, if  $\bar{u}$  and  $\underline{v}$  are finite for all  $x \in \bar{\Omega}$  and  $t_* < t \leq T$ , then they are respectively a viscosity subsolution and a viscosity supersolution of (1.2.12) on this time interval.

*Proof.* When  $x_0 \in \Omega$ , since we can find in  $\Omega$  a  $\delta$ -neighborhood of  $x_0$ , the proof follows from [88, Proposition 3.3]. Therefore we shall focus on the case when  $x_0 \in \partial\Omega$ . We give the proof for  $\bar{u}$ . The argument for  $\underline{v}$  is entirely parallel, relying on Proposition 1.4.5. Consider a smooth function  $\phi$  such that  $\bar{u} - \phi$  has a local maximum at  $(x_0, t_0)$ . Adding a constant, we can assume  $\bar{u}(x_0, t_0) = \phi(x_0, t_0)$ . Replacing  $\phi$  by  $\phi + \|x - x_0\|^4 + |t - t_0|^2$  if necessary, we can assume that the local maximum is strict, i.e. that

$$\bar{u}(x, t) < \phi(x, t) \quad \text{for } 0 < \|(x, t) - (x_0, t_0)\| \leq r, \quad (1.3.3)$$

for some  $r > 0$ . By the definition of  $\bar{u}$ , there exist sequences  $\varepsilon_k, \tilde{y}_k, \tilde{t}_k = T - \tilde{N}_k \varepsilon_k^2$  such that

$$\tilde{y}_k \rightarrow x_0, \quad \tilde{t}_k \rightarrow t_0, \quad u^{\varepsilon_k}(\tilde{y}_k, \tilde{t}_k) \rightarrow \bar{u}(x_0, t_0).$$

Let  $y_k$  and  $t_k = T - N_k \varepsilon_k^2$  satisfying

$$(u^{\varepsilon_k} - \phi)(y_k, t_k) \geq \sup_{\|(x, t) - (x_0, t_0)\| \leq r} (u^{\varepsilon_k} - \phi)(x, t) - \varepsilon_k^3.$$

Notice that since  $u^{\varepsilon_k}$  is defined only at discrete times, the sup is taken only over such times. Evidently,

$$(u^{\varepsilon_k} - \phi)(y_k, t_k) \geq (u^{\varepsilon_k} - \phi)(\tilde{y}_k, \tilde{t}_k) - \varepsilon_k^3$$

and the right-hand side tends to 0 as  $k \rightarrow +\infty$ . It follows using (1.3.3) that

$$(y_k, t_k) \rightarrow (y_0, t_0) \quad \text{and} \quad u^{\varepsilon_k}(y_k, t_k) \rightarrow \bar{u}(x_0, t_0),$$

as  $k \rightarrow +\infty$ . Setting  $\xi_k = u^{\varepsilon_k}(y_k, t_k) - \phi(y_k, t_k)$ , we also have by construction that

$$\xi_k \rightarrow 0 \text{ and } u^{\varepsilon_k}(x, t) \leq \phi(x, t) + \xi_k + \varepsilon_k^3 \quad \text{whenever } t = T - n_k \varepsilon_k^2 \text{ and } \|(x, t) - (x_0, t_0)\| \leq r. \quad (1.3.4)$$

Now we use the dynamic programming inequality (1.2.24) at  $(y_k, t_k)$ , which can be written concisely as

$$u^{\varepsilon_k}(y_k, t_k) \leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \{u^{\varepsilon_k}(y_k + \Delta x, t_k + \varepsilon_k^2) - \Delta z\},$$

with the convention

$$\Delta z = p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon_k^2 \mathcal{F}(t_k, y_k, u^{\varepsilon_k}(y_k, t_k), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(y_k + \Delta x).$$

Using the definition of  $\xi_k$ , (1.3.4), and the fact that  $\Delta x$  is bounded by a positive power of  $\varepsilon$ , we conclude that

$$\phi(y_k, t_k) + \xi_k \leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \{ \phi(y_k + \Delta x, t_k + \varepsilon_k^2) + \xi_k + \varepsilon_k^3 - \Delta z \}, \quad (1.3.5)$$

when  $k$  is sufficiently large. Dropping  $\xi_k$  from both sides of (1.3.5), we deduce, by introducing the operator  $S_\varepsilon$  defined by (1.4.1), that

$$\phi(y_k, t_k) \leq S_\varepsilon[y_k, t_k, u^{\varepsilon_k}(y_k, t_k), \phi(\cdot, t_k + \varepsilon_k^2)] + o(\varepsilon_k^2). \quad (1.3.6)$$

According to the consistency estimates provided by Proposition 1.4.10, we shall introduce four sets  $(A_i)_{1 \leq i \leq 4}$  respectively defined by

$$\begin{aligned} A_1 &:= \left\{ k \in \mathbb{N} : \text{dist}(y_k) \leq \varepsilon_k^{1-\alpha} \text{ and } M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \geq \frac{4}{3} \|D^2\phi(y_k, t_k + \varepsilon_k^2)\| \varepsilon_k^{1-\alpha} \right\}, \\ A_2 &:= \left\{ k \in \mathbb{N} : \varepsilon_k^{1-\alpha} - \varepsilon_k^\rho \leq \text{dist}(y_k) \leq \varepsilon_k^{1-\alpha} \text{ and } M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \leq \frac{4}{3} \|D^2\phi(y_k, t_k + \varepsilon_k^2)\| \varepsilon_k^{1-\alpha} \right\} \\ &\quad \cup \left\{ k \in \mathbb{N} : \text{dist}(y_k) \geq \varepsilon_k^{1-\alpha} \right\}, \\ A_3 &:= \left\{ k \in \mathbb{N} : \text{dist}(y_k) \leq \varepsilon_k^{1-\alpha} - \varepsilon_k^\rho \text{ and } -\varepsilon_k^{1-\alpha-\kappa} \leq M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \leq \frac{4}{3} \|D^2\phi(y_k, t_k + \varepsilon_k^2)\| \varepsilon_k^{1-\alpha} \right\}, \\ A_4 &:= \left\{ k \in \mathbb{N} : \text{dist}(y_k) \leq \varepsilon_k^{1-\alpha} - \varepsilon_k^\rho \text{ and } M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \leq -\varepsilon_k^{1-\alpha-\kappa} \right\}, \end{aligned}$$

where  $\rho$  and  $\kappa$  are defined in Section 1.4.1.2 by (1.4.23) and (1.4.24) and satisfy  $0 < \kappa < 1 - \alpha < \rho < 1$ . Since  $\cup_{1 \leq i \leq 4} A_i = \mathbb{N}$ , at least one set among  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  is necessarily unbounded. We shall consider these four cases.

- If  $A_1$  is unbounded, up to a subsequence, we may assume that  $A_1 = \mathbb{N}$ . Taking the limit  $k \rightarrow +\infty$ , we deduce that  $\liminf_{k \rightarrow +\infty} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \geq 0$ . Since  $M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \rightarrow h(x_0) - D\phi(x_0, t_0) \cdot n(x_0)$  by Lemma 1.3.3, it follows in the limit  $k \rightarrow \infty$  that

$$D\phi(x_0, t_0) \cdot n(x_0) - h(x_0) \leq 0. \quad (1.3.7)$$

We can notice this result also holds through (1.3.6). We can apply the second alternative given by (1.4.60) in Proposition 1.4.16 to evaluate the right-hand side of (1.3.6). This gives

$$\phi(y_k, t_k) - \phi(y_k, t_k + \varepsilon_k^2) \leq 3\varepsilon_k^{1-\alpha} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] + C\varepsilon_k^2(1 + |u^{\varepsilon_k}(y_k, t_k)|) + o(\varepsilon_k^2),$$

where  $C$  depends only on  $\|h\|_{L^\infty}$  and  $\|D\phi(\cdot, t_k + \varepsilon_k^2)\|_{C_b^1(\bar{\Omega} \cap B(y_k, \varepsilon_k^{1-\alpha}))}$ . Since for  $k$  large enough,

$$\|D\phi(\cdot, t_k + \varepsilon_k^2)\|_{C_b^1(\bar{\Omega} \cap B(y_k, \varepsilon_k^{1-\alpha}))} \leq \sup_{|t-t_0| \leq r} \|D\phi(\cdot, t)\|_{C_b^1(\bar{\Omega} \cap B(x_0, r))},$$

we can suppose that  $C$  depends only on  $\|h\|_{L^\infty}$  and this sup, which is finite (since  $\phi$  is smooth) and independent of  $k$ . By smoothness of  $\phi$  we have

$$-\varepsilon_k^2 \partial_t \phi(y_k, t_k) + o(\varepsilon_k^2) - C(1 + |u^{\varepsilon_k}(y_k, t_k)|) \varepsilon_k^2 \leq 3\varepsilon_k^{1-\alpha} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)].$$

By dividing by  $\varepsilon_k^{1-\alpha}$  we obtain

$$-\varepsilon_k^{1+\alpha} \left( \partial_t \phi(y_k, t_k) - C(1 + |u^{\varepsilon_k}(y_k, t_k)|) \right) + o(\varepsilon_k^{1+\alpha}) \leq 3M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)].$$

The sequences  $(u^{\varepsilon_k}(y_k, t_k))_{k \in \mathbb{N}}$  and  $(\partial_t \phi(y_k, t_k))_{k \in \mathbb{N}}$  are respectively bounded by definition of  $\bar{u}(x_0, t_0)$  and smoothness of  $\phi$ . By passing to the limit on  $k$ ,  $\liminf_{k \rightarrow +\infty} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \geq 0$ . By Lemma 1.3.3, we know that  $M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \rightarrow h(x_0) - D\phi(x_0, t_0) \cdot n(x_0)$  and (1.3.7) is retrieved.

- If  $A_2$  is unbounded, up to a subsequence, we may assume that  $A_2 = \mathbb{N}$ . We can apply Proposition 1.4.10 case (ii) to evaluate the right-hand side of (1.3.6). This gives

$$\phi(y_k, t_k) \leq \phi(y_k, t_k + \varepsilon_k^2) - \varepsilon_k^2 \mathcal{F}(t_k, y_k, u^{\varepsilon_k}(y_k, t_k), D\phi(y_k, t_k + \varepsilon_k^2), D^2\phi(y_k, t_k + \varepsilon_k^2)) + o(\varepsilon_k^2).$$

By smoothness of  $\phi$  and Lipschitz continuity of  $\mathcal{F}$  with respect to  $p$  and  $\Gamma$ , we obtain

$$\phi(y_k, t_k) - \phi(y_k, t_k + \varepsilon_k^2) \leq -\varepsilon_k^2 \mathcal{F}(t_k, y_k, u^{\varepsilon_k}(y_k, t_k), D\phi(y_k, t_k), D^2\phi(y_k, t_k)) + o(\varepsilon_k^2).$$

It follows in the limit  $k \rightarrow \infty$  that

$$\partial_t \phi(x_0, t_0) - \mathcal{F}(t_0, x_0, \bar{u}(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0. \quad (1.3.8)$$

- If  $A_3$  is unbounded, up to a subsequence, we may assume that  $A_3 = \mathbb{N}$ . By passing to the limit on  $k$ , we have that  $M_{\varepsilon_k}^{y_k, t_k + \varepsilon_k^2}[\phi]$  tends to zero when  $\varepsilon_k$  tends to zero. Since  $M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \rightarrow h(x_0) - D\phi(x_0, t_0) \cdot n(x_0)$  by Lemma 1.3.3, it follows in the limit  $k \rightarrow \infty$  that  $D\phi(x_0, t_0) \cdot n(x_0) - h(x_0) = 0$ .
- If  $A_4$  is unbounded, up to a subsequence, we may assume that  $A_4 = \mathbb{N}$ . Hence, taking the limit  $k \rightarrow +\infty$ , we have

$$\limsup_{k \rightarrow +\infty} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \leq 0. \quad (1.3.9)$$

Moreover, by applying the fourth alternative given by (1.4.60) in Proposition 1.4.16 to evaluate the right-hand side of (1.3.6), we obtain

$$\phi(y_k, t_k) \leq \phi(y_k, t_k + \varepsilon_k^2) + \frac{1}{4} \varepsilon_k^\rho M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] + C \varepsilon_k^2 (1 + |u^{\varepsilon_k}(y_k, t_k)|) + o(\varepsilon_k^2),$$

where  $C$  depends only on  $\|h\|_{L^\infty}$  and  $\sup_{|t-t_0| \leq r} \|D\phi(\cdot, t)\|_{C_b^1(\overline{\Omega} \cap B(x_0, r))}$  by the same arguments used above for  $A_1$ . By smoothness of  $\phi$  we have

$$-\varepsilon_k^2 \partial_t \phi(y_k, t_k) + o(\varepsilon_k^2) - C(1 + |u^{\varepsilon_k}(y_k, t_k)|) \varepsilon_k^2 \leq \frac{1}{4} \varepsilon_k^\rho M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)].$$

By dividing by  $\varepsilon_k^\rho$  we get

$$-\varepsilon_k^{2-\rho} \left( \partial_t \phi(y_k, t_k) - C(1 + |u^{\varepsilon_k}(y_k, t_k)|) \right) + o(\varepsilon_k^{2-\rho}) \leq \frac{1}{4} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)].$$

The sequences  $(u^{\varepsilon_k}(y_k, t_k))_{k \in \mathbb{N}}$  and  $(\partial_t \phi(y_k, t_k))_{k \in \mathbb{N}}$  are respectively bounded by definition of  $\bar{u}(x_0, t_0)$  and smoothness of  $\phi$ . By passing to the limit as  $k \rightarrow +\infty$ , we have

$$\liminf_{k \rightarrow +\infty} M_{\varepsilon_k}^{y_k}[\phi(\cdot, t_k + \varepsilon_k^2)] \geq 0.$$

By comparing this inequality with (1.3.9) and using Lemma 1.3.3, we deduce that

$$D\phi(x_0, t_0) \cdot n(x_0) - h(x_0) = 0.$$

Moreover, we can also apply Lemma 1.4.9 since  $\varepsilon_k^{1-\alpha} \ll \varepsilon_k^{1-\alpha-\kappa}$ . By the same manipulations as those done for the set  $A_2$ , the inequality (1.3.8) holds also true.

Thus  $\bar{u}$  is a viscosity subsolution at  $(x_0, t_0)$ .

We turn now to case (iii), i.e. the case  $t_0 = T$ . If  $x_0 \in \Omega$ , the analysis led in [88, Proposition 3.3] gives the result. It remains to study  $\bar{u}$  on the boundary. We want to show that  $\bar{u}(\cdot, T) = g$  is also satisfied on  $\partial\Omega$ . By the definition of  $\bar{u}$  given by (1.2.26) and considering a particular sequence  $(\varepsilon_k, x_k, t_k = T)_{k \in \mathbb{N}}$  which converges to  $(0, x_0, T)$ , it is clear that  $\bar{u}(\cdot, T) \geq g$  on  $\partial\Omega$  (using the continuity of  $g$  and the fact that each  $u^\varepsilon$  has final value  $g$ ). If this sequence realizes the sup, we have in fact the equality. The preceding argument can still be used provided  $t_k < T$  for all sufficiently large  $k$ . Thus, considering the different possibilities according to  $t_k < T$  or  $t_k = T$  and also on  $x_k \in \Omega$  or  $x_k \in \partial\Omega$ , we know that for any smooth  $\phi$  such that  $\bar{u} - \phi$  has a local maximum at  $(x_0, T)$ ,

either  $\bar{u}(x_0, T) = g(x_0)$  or else

$$\max(\partial_t \phi(x_0, T) - \mathcal{F}(t_0, x_0, \bar{u}(x_0, T), D\phi(x_0, T), D^2\phi(x_0, T)), h(x_0) - D\phi(x_0, T) \cdot n(x_0)) \geq 0. \quad (1.3.10)$$

Moreover this statement applies not only at the given point  $x_0$ , but also at any point nearby.

Now consider the functions

$$\psi(x, t) = \bar{u}(x, t) - \frac{\|x - x_0\|^2}{\eta} - \frac{T - t}{\mu} + K \operatorname{dist}(x)$$

and

$$\phi(x, t) = \frac{\|x - x_0\|^2}{\eta} + \frac{T - t}{\mu} - K \operatorname{dist}(x), \quad (1.3.11)$$



where the parameters  $\eta, \mu$  are small and positive and  $K = \|h\|_{L^\infty} + 1$ . Suppose  $\bar{u}$  is uniformly bounded on the closed half-ball  $\{\|(x, t) - (x_0, T)\| \leq r, t \leq T\}$  and let  $\psi$  attain its maximum on this half-ball at  $(x_{\eta, \mu}, t_{\eta, \mu})$ . We assume  $r$  is small enough such that  $d$  is  $C^2$  on this half-ball so that  $\phi$  can be taken as a test function. We clearly have

$$\bar{u}(x_{\eta, \mu}, t_{\eta, \mu}) + Kd(x_{\eta, \mu}) \geq \psi(x_{\eta, \mu}, t_{\eta, \mu}) \geq \psi(x_0, T) = \bar{u}(x_0, T). \quad (1.3.12)$$

By plugging the expression of  $\psi(x_{\eta, \mu}, t_{\eta, \mu})$  in the right-hand side of inequality (1.3.12), we obtain

$$0 \leq \frac{\|x_{\eta, \mu} - x_0\|^2}{\eta} + \frac{T - t_{\eta, \mu}}{\mu} \leq \bar{u}(x_{\eta, \mu}, t_{\eta, \mu}) - \bar{u}(x_0, T) + Kd(x_{\eta, \mu}). \quad (1.3.13)$$

Since  $\bar{u}$  is bounded on the half-ball and  $x_{\eta, \mu}$  belongs to the half ball for all  $\eta$  and  $\mu$ , the right-hand side of (1.3.13) is bounded independently of  $\eta, \mu$ , which yields

$$(x_{\eta, \mu}, t_{\eta, \mu}) \rightarrow (x_0, T) \quad \text{as } \eta, \mu \rightarrow 0. \quad (1.3.14)$$

By using the upper semicontinuity of  $\bar{u}$  and taking the limit (1.3.14) in (1.3.12), we get

$$\bar{u}(x_{\eta, \mu}, t_{\eta, \mu}) \rightarrow \bar{u}(x_0, T) \quad \text{as } \eta, \mu \rightarrow 0. \quad (1.3.15)$$

By combining (1.3.14) and (1.3.15), we finally obtain by (1.3.13) that

$$\frac{\|x_{\eta, \mu} - x_0\|^2}{\eta} + \frac{T - t_{\eta, \mu}}{\mu} \rightarrow 0 \quad \text{as } \eta, \mu \rightarrow 0. \quad (1.3.16)$$

If  $t_{\eta, \mu} < T$  and  $x_{\eta, \mu} \in \Omega$  then part (i) of Proposition 1.3.4 applied to  $\phi$  defined by (1.3.11) assures us that

$$-\frac{1}{\mu} - \mathcal{F}(t_{\eta, \mu}, x_{\eta, \mu}, \bar{u}(x_{\eta, \mu}, t_{\eta, \mu}), 2\frac{x_{\eta, \mu} - x_0}{\eta} - KDd(x_{\eta, \mu}), \frac{2}{\eta}I - KD^2d(x_{\eta, \mu})) \geq 0. \quad (1.3.17)$$

Since  $\mathcal{F}$  is continuous, for any  $\eta > 0$  there exists  $\mu > 0$  such that (1.3.17) cannot happen. Restricting our attention to such choices of  $\eta$  and  $\mu$ , it remains to examine two situations: on the one hand  $t_{\eta, \mu} < T$  and  $x_{\eta, \mu} \in \partial\Omega$  and on the other hand  $t_{\eta, \mu} = T$ . Arguing by contradiction, let us assume that  $t_{\eta, \mu} < T$  and  $x_{\eta, \mu} \in \partial\Omega$ . By the Taylor expansion on the distance function close to  $x_0$ , we have

$$\text{dist}(x) = \text{dist}(x_0) + D\text{dist}(x_0) \cdot (x - x_0) + O(\|x - x_0\|^2).$$

By using that  $x_0$  and  $x_{\eta, \mu}$  are on the boundary  $\partial\Omega$ ,  $d(x_0) = d(x_{\eta, \mu}) = 0$  and  $Dd(x_0) = -n(x_0)$ , this relation reduces to

$$n(x_0) \cdot (x_{\eta, \mu} - x_0) = O(\|x_{\eta, \mu} - x_0\|^2). \quad (1.3.18)$$

By combining (1.3.16) and (1.3.18), we compute

$$\begin{aligned} D\phi(x_{\eta, \mu}, t_{\eta, \mu}) \cdot n(x_0) &= \frac{2}{\eta}(x_{\eta, \mu} - x_0) \cdot n(x_0) - KD\text{dist}(x_{\eta, \mu}) \cdot n(x_0) \\ &= O\left(\frac{\|x_{\eta, \mu} - x_0\|^2}{\eta}\right) + Kn(x_{\eta, \mu}) \cdot n(x_0) \rightarrow K, \quad \text{as } \eta, \mu \rightarrow 0. \end{aligned}$$

By smoothness of  $\phi$  and continuity of  $n$  on  $\partial\Omega$ , we deduce that  $D\phi(x_{\eta, \mu}, T) \cdot n(x_{\eta, \mu}) \rightarrow \|h\|_{L^\infty} + 1 > h(x_{\eta, \mu})$  which denies the second alternative proposed at (1.3.10). As a result, the only remaining possibility for (1.3.10) is  $\bar{u}(x_{\eta, \mu}, T) = g(x_{\eta, \mu})$ . By continuity of  $g$ , it follows in the limit  $\eta, \mu \rightarrow 0$  that  $\bar{u}(x_0, T) = g(x_0)$ , as asserted.  $\square$

**Remark 1.3.5.** In the proof of the convergence at the final-time in Theorem 1.3.4, we needed in a essential way that the domain was assumed to be at least  $C^2$ . More precisely, in this case, since the distance function  $d$  is  $C^2$  in a neighborhood of the boundary, it allows us to take  $\phi$  given by (1.3.11) as a test function.

### 1.3.3 The elliptic case

We turn now to the stationary setting discussed in Section 1.2.3. As in the time-dependent setting, our convergence result depends on the fundamental consistency result Proposition 1.4.18. So we require that the parameters  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15), and that  $\mathcal{F}(x, z, p, \Gamma)$  satisfy not only the monotonicity condition (1.1.4) but also the Lipschitz continuity and growth conditions (1.1.9)–(1.1.10). To prove that  $U^\varepsilon$  is well defined, we require that the interest rate  $\lambda$  be large enough, condition (1.1.8), and that  $h$  be uniformly bounded. Finally, concerning the parameters  $m$  and  $M$  and the function  $\psi$  associated to the termination of the game, we assume that  $\psi \in C^2(\overline{\Omega})$  fulfills  $\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1$  on  $\partial\Omega$ ,  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ ,  $\chi = m + \|\psi\|_{L^\infty(\overline{\Omega})} + \psi$  and  $M$  is sufficiently large. These hypotheses ensure us the availability of the dynamic programming inequalities stated in Proposition 3.3.1.

**Proposition 1.3.6.** *Suppose  $\mathcal{F}, g, \lambda$  and  $\alpha, \beta, \gamma, m, M, \psi$  satisfy the hypotheses just listed (from which it follows that  $\underline{v}$  and  $\overline{u}$  are bounded away from  $\pm M$  for all  $x \in \overline{\Omega}$ ). Then  $\overline{u}$  is a viscosity subsolution and  $\underline{v}$  is a viscosity supersolution of (1.2.54) in  $\overline{\Omega}$ . More specifically:*

- if  $x_0 \in \Omega$  then each of  $\overline{u}$  and  $\underline{v}$  satisfies part (E1) of relevant half of Definition 1.3.2 at  $x_0$ , and
- if  $x_0 \in \partial\Omega$  then each of  $\overline{u}$  and  $\underline{v}$  satisfies part (E2) of relevant half of Definition 1.3.2 at  $x_0$ .

*Proof.* When  $x_0 \in \Omega$ , the proof is similar to that of Theorem 1.3.4. Therefore we shall focus on the case when  $x_0 \in \partial\Omega$ . We give the proof for  $\overline{u}$ , the arguments for  $\underline{v}$  being similar and even easier due to fewer cases to distinguish. Consider a smooth function  $\phi$  such that  $\overline{u} - \phi$  has local maximum on  $\overline{\Omega}$  at  $x_0 \in \partial\Omega$ . We may assume that  $\langle D\phi(x_0), n(x_0) \rangle > h(x_0)$  since otherwise the assertion is trivial. Adjusting  $\phi$  if necessary, we can assume that  $\overline{u}(x_0) = \phi(x_0)$  and that the local maximum is strict, i.e.

$$\overline{u}(x) < \phi(x) \quad \text{for } x \in \overline{\Omega} \cap \{0 < \|x - x_0\| \leq r\}, \quad (1.3.19)$$

for some  $r > 0$ . By the definition of  $\overline{u}$  given by (1.2.65), there exist sequences  $\varepsilon_k > 0$  and  $\tilde{y}_k \in \overline{\Omega}$  such that

$$\tilde{y}_k \rightarrow x_0, \quad u^{\varepsilon_k}(\tilde{y}_k) \rightarrow \overline{u}(x_0).$$

We may choose  $y_k \in \overline{\Omega}$  such that  $(u^{\varepsilon_k} - \phi)(y_k) \geq \sup_{\overline{\Omega} \cap \{\|x - x_0\| \leq r\}} (u^{\varepsilon_k} - \phi)(x) - \varepsilon_k^3$ . Evidently

$$(u^{\varepsilon_k} - \phi)(y_k) \geq (u^{\varepsilon_k} - \phi)(\tilde{y}_k) - \varepsilon_k^3$$

and the right-hand side tends to 0 as  $k \rightarrow \infty$ . It follows using (1.3.19) that

$$y_k \rightarrow x_0 \quad \text{and} \quad u^{\varepsilon_k}(y_k) \rightarrow \overline{u}(x_0),$$

as  $k \rightarrow \infty$ . Setting  $\xi_k = (u^{\varepsilon_k} - \phi)(y_k)$ , we also have by construction that

$$\xi_k \rightarrow 0 \quad \text{and} \quad u^{\varepsilon_k}(x) \leq \phi(x) + \xi_k - \varepsilon_k^3 \quad \text{whenever } x \in \overline{\Omega} \text{ with } \|x - x_0\| \leq r. \quad (1.3.20)$$

We now use the dynamic programming inequality (1.2.62) at  $y_k$ , which can be written concisely as

$$u^{\varepsilon_k}(y_k) \leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left\{ e^{-\lambda \varepsilon_k^2} u^{\varepsilon_k}(y_k + \Delta x) - \delta_k \right\},$$

with the convention

$$\delta_k = p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon_k^2 \mathcal{F}(x, u^{\varepsilon_k}(x), p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x).$$

By the rule (1.2.18) of the game, for every move  $\Delta \hat{x}$  decided by Mark, the point  $y_k + \Delta x$  belongs to  $\overline{\Omega}$ . Combining this observation with (1.3.20) and the definition of  $\xi_k$  we conclude that

$$\phi(y_k) + \xi_k \leq \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left\{ e^{-\lambda \varepsilon_k^2} [\phi(y_k + \Delta x) + \xi_k - \varepsilon_k^3] - \delta_k \right\}.$$

We may replace  $e^{-\lambda\varepsilon_k^2}$  by  $1 - \lambda\varepsilon_k^2$  and  $e^{-\lambda\varepsilon_k^2}\xi_k$  by  $\xi_k$  while making an error which is only  $o(\varepsilon^2)$  using the fact that  $\xi_k \rightarrow 0$ . Dropping  $\xi_k$  from both sides, we conclude that

$$\phi(y_k) \leq \sup_{p, \Gamma} \inf_{\Delta \bar{x}} \left( e^{-\lambda\varepsilon_k^2} \phi(y_k + \Delta x) - \delta_k \right) + o(\varepsilon_k^2).$$

We can evaluate the right-hand side using Proposition 1.4.18 case (ii) for  $k$  large enough. We introduce  $\rho$  and  $\kappa$  defined in Section 1.4.1.2 by (1.4.23) and (1.4.24) and satisfying  $0 < \kappa < 1 - \alpha < \rho < 1$ . If we may assume, up to a subsequence, that for all  $k$  large enough, on the one hand  $\text{dist}(y_k) \geq \varepsilon_k^{1-\alpha}$  or on the one hand  $\varepsilon_k^{1-\alpha} - \varepsilon_k^\rho \leq \text{dist}(y_k) \leq \varepsilon_k^{1-\alpha}$  and  $M_{\varepsilon_k}^{y_k}[\phi] \leq \frac{4}{3} \|D^2\phi(y_k)\| \varepsilon_k^{1-\alpha}$ , we can apply Proposition 1.4.18 case (ii) to evaluate the right-hand side

$$0 \leq -\varepsilon_k^2 f(y_k, u^{\varepsilon_k}(y_k), D\phi(y_k), D^2\phi(y_k)) - \varepsilon_k^2 \lambda u^{\varepsilon_k}(y_k) + o(\varepsilon_k^2).$$

By passing to the limit  $k \rightarrow +\infty$ , we get the required inequality in the viscosity sense. Otherwise, recall that  $\langle D\phi(x_0), n(x_0) \rangle > h(x_0)$ . By Lemma 1.3.3, we have

$$M_{\varepsilon_k}^{y_k}[\phi] \rightarrow h(x_0) - \langle D\phi(x_0), n(x_0) \rangle < 0, \quad (1.3.21)$$

and the condition

$$M_{\varepsilon_k}^{y_k}[\phi] \leq -\varepsilon_k^{1-\alpha-\kappa} \quad (1.3.22)$$

is satisfied for all  $k$  sufficiently large. Therefore, up to a subsequence, it remains to consider a sequence  $(y_k, \varepsilon_k)_{k \in \mathbb{N}}$  satisfying both  $\text{dist}(y_k) \leq \varepsilon_k^{1-\alpha} - \varepsilon_k^\rho$  and (1.3.22). The last part of Proposition 1.4.19 can be applied and we get by (1.4.71) that there exists a constant  $C$  depending only on  $M$ ,  $\|h\|_{L^\infty}$  and  $\|D\phi\|_{C_b^1(\bar{\Omega}) \cap B(y_k, \varepsilon_k^{1-\alpha})}$  such that

$$0 \leq \frac{1}{4} \varepsilon_k^\rho M_{\varepsilon_k}^{y_k}[\phi] + C\varepsilon_k^2 - \lambda\varepsilon_k^2 \phi(y_k) + o(\varepsilon_k^2),$$

recalling that  $(\varepsilon_k^{1-\alpha} - \text{dist}(y_k)) \geq \varepsilon_k^\rho$  and  $M_{\varepsilon_k}^{y_k}[\phi] < 0$ . By dividing by  $\varepsilon_k^\rho$ , it follows that

$$-\varepsilon_k^{2-\rho}(C - \lambda\phi(y_k)) + o(\varepsilon_k^{2-\rho}) \leq \frac{1}{4} M_{\varepsilon_k}^{y_k}[\phi].$$

The sequence  $(\phi(y_k))_{k \in \mathbb{N}}$  is bounded by smoothness of  $\phi$ . Since  $\|D\phi\|_{C_b^1(\bar{\Omega}) \cap B(y_k, \varepsilon_k^{1-\alpha})} \leq \|D\phi\|_{C_b^1(\bar{\Omega}) \cap B(x_0, r)}$  holds for  $k$  large enough, we can assume that  $C$  is independent of  $k$  depending only on  $\|D\phi\|_{C_b^1(\bar{\Omega}) \cap B(x_0, r)}$ ,  $M$  and  $\|h\|_{L^\infty}$ . Taking the limit as  $k \rightarrow +\infty$ , we deduce that

$$\liminf_{k \rightarrow \infty} M_{\varepsilon_k}^{y_k}[\phi] \geq 0,$$

which is a contradiction with (1.3.21). Thus  $\bar{u}$  is a viscosity subsolution at  $x_0$ .  $\square$

## 1.4 Consistency

A numerical scheme is said to be consistent if every smooth solution of the PDE satisfies it modulo an error that tends to zero with the step size. It is the idea of the argument used in [88]. In our case, we must understand how the Neumann condition interferes with the estimates proposed in [88, Section 4]. The essence of our formal argument in Section 1.2.2.1 was that the Neumann condition term is predominant compared to the PDE term at the boundary and produces a degeneracy in the consistency estimate. The present section clarifies the connection between our formal argument and the consistency of the game, by discussing consistency in more conventional terms. The main point is presented in Propositions 1.4.5 and 1.4.10. In order to explain very precisely how the consistency estimate of [88, Section 4] degenerates, we establish the consistency of our game as a numerical scheme by focusing on different cases according to the values of the quantities  $m_\varepsilon^x[\phi]$  and  $M_\varepsilon^x[\phi]$  defined by (1.3.1)–(1.3.2) and the distance  $\text{dist}(x)$  to the boundary  $\partial\Omega$ .

### 1.4.1 The parabolic case

Consider the game discussed in Section 1.2.2 for solving  $-u_t + \mathcal{F}(t, x, u, Du, D^2u) = 0$  in  $\Omega$  with final-time data  $u(x, T) = g(x)$  for  $x \in \overline{\Omega}$  and boundary condition  $\frac{\partial u}{\partial n}(x, t) = h(x)$  for  $x \in \partial\Omega, t \in (0, T)$ . The dynamic programming principles (1.2.24)–(1.2.25) can be written as

$$\begin{aligned} u^\varepsilon(x, t) &\leq S_\varepsilon [x, t, u^\varepsilon(x, t), u^\varepsilon(\cdot, t + \varepsilon^2)], \\ v^\varepsilon(x, t) &\geq S_\varepsilon [x, t, v^\varepsilon(x, t), v^\varepsilon(\cdot, t + \varepsilon^2)], \end{aligned}$$

where  $S_\varepsilon [x, t, z, \phi]$  is defined for any  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$  and  $t \leq T$  and any continuous function  $\phi: \overline{\Omega} \rightarrow \mathbb{R}$  by

$$\begin{aligned} S_\varepsilon [x, t, z, \phi] &= \max_{p, \Gamma} \min_{\Delta \hat{x}} \left[ \phi(x + \Delta x) \right. \\ &\quad \left. - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right], \end{aligned} \quad (1.4.1)$$

subject to the usual constraints  $\|p\| \leq \varepsilon^{-\beta}$ ,  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ ,  $\|\Delta \hat{x}\| \leq \varepsilon^{1-\alpha}$  and  $\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta \hat{x}) - x$ . The operator  $S_\varepsilon$  clearly satisfies the three following properties:

- For all  $\phi \in C(\overline{\Omega})$ ,  $S_0 [x, t, z, \phi] = \phi(x)$ .
- $S_\varepsilon$  is monotone, i.e. if  $\phi_1 \leq \phi_2$ , then  $S_\varepsilon [x, t, z, \phi_1] \leq S_\varepsilon [x, t, z, \phi_2]$ .
- For all  $\phi \in C(\overline{\Omega})$  and  $c \in \mathbb{R}$ ,

$$S_\varepsilon [x, t, z, c + \phi] = c + S_\varepsilon [x, t, z, \phi]. \quad (1.4.2)$$

Fixing  $x, t, z$  and a smooth function  $\phi$ , a Taylor expansion shows that for any  $\|\Delta x\| \leq \varepsilon^{1-\alpha}$ ,

$$\phi(x + \Delta x) = \phi(x) + D\phi(x) \cdot \Delta x + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle + O(\varepsilon^{3-3\alpha}).$$

Since  $\alpha < 1/3$  by hypothesis,  $\varepsilon^{3-3\alpha} = o(\varepsilon^2)$ . By rearranging the terms, we compute

$$\begin{aligned} \phi(x + \Delta x) &- \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \\ &= \phi(x) + (D\phi(x) - p) \cdot \Delta \hat{x} + \|\Delta \hat{x} - \Delta x\| [h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)] \\ &\quad + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) + o(\varepsilon^2), \end{aligned}$$

since the outward normal can be expressed by  $n(x + \Delta x) = -\frac{\Delta x - \Delta \hat{x}}{\|\Delta \hat{x} - \Delta x\|}$  if  $x + \Delta \hat{x} \notin \Omega$  and the move  $\Delta x$  can be decomposed as  $\Delta x = \Delta \hat{x} + (\Delta x - \Delta \hat{x})$ . Thus, we shall examine

$$\begin{aligned} S_\varepsilon [x, t, z, \phi] - \phi(x) &= \max_{p, \Gamma} \min_{\Delta \hat{x}} \left[ (D\phi(x) - p) \cdot \Delta \hat{x} + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle \right. \\ &\quad \left. + \|\Delta \hat{x} - \Delta x\| \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\} - \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right] + o(\varepsilon^2). \end{aligned} \quad (1.4.3)$$

#### 1.4.1.1 Preliminary geometric lemmas

This subsection is devoted to some geometric properties of the game that will be useful to show consistency in Section 1.4.1.2. We start by some estimates, involving the geometric conditions on the domain, about the moves  $\Delta \hat{x}$  decided by Mark.

**Lemma 1.4.1.** *Suppose that  $\Omega$  is a  $C^2$ -domain satisfying the uniform exterior ball condition for a certain  $r > 0$ . Then, for all  $0 < \varepsilon < r^{\frac{1}{1-\alpha}}$  and for all  $\Delta\hat{x}$  constrained by (1.2.17) and determining  $\Delta x$  by (1.2.21), we have*

$$\|\Delta\hat{x} - \Delta x\| \leq \varepsilon^{1-\alpha} - \text{dist}(x) \quad \text{and} \quad \|\Delta x\| \leq 2\varepsilon^{1-\alpha} - \text{dist}(x). \quad (1.4.4)$$

*Proof.* Let us prove the first inequality, the second following immediately by the triangle inequality. If the point  $\hat{x} = x + \Delta\hat{x}$  belongs to  $\overline{\Omega}$ ,  $\Delta x = \Delta\hat{x}$  and the result is obvious. Supposing now  $\hat{x}$  does not belong to  $\overline{\Omega}$ , the set  $S = [x, \hat{x}] \cap \partial\Omega$  is not empty and we can consider a point  $x_\partial \in S$ . By the rule of the game, we have  $\|x - \hat{x}\| = \|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ . Since  $x_\partial \in \partial\Omega$  by construction, it is clear that  $\|x - x_\partial\| \geq \text{dist}(x)$ . We deduce that

$$\|x_\partial - \hat{x}\| = \|x - \hat{x}\| - \|x_\partial - x\| \leq \varepsilon^{1-\alpha} - \text{dist}(x).$$

By the uniform exterior ball condition, the orthogonal projection on  $\overline{\Omega}$  is well-defined on  $\Omega(\varepsilon^{1-\alpha}) \subset \Omega(r)$ . By property of the orthogonal projection and since  $\hat{x} \notin \overline{\Omega}$ , we can write

$$\|\Delta\hat{x} - \Delta x\| = \inf_{y \in \overline{\Omega}} \|y - \hat{x}\| = \inf_{y \in \partial\Omega} \|y - \hat{x}\| \leq \|x_\partial - \hat{x}\|,$$

which gives directly the first estimate in (1.4.4).  $\square$

The following lemma uses the crucial geometric fact that  $\Omega$  satisfies the interior ball condition introduced in Definition 1.1.2 for which there is no neck pitching for  $\varepsilon$  sufficiently small.

**Lemma 1.4.2.** *Let  $\sigma > 1 - \alpha$  and  $B > 0$ . Suppose that  $\Omega \subset \mathbb{R}^d$  is a domain with  $C^2$ -boundary  $\partial\Omega$  and satisfies the uniform interior ball condition. Then, for all possible moves  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$  such that  $\|\Delta\hat{x} + \varepsilon^{1-\alpha}n(\bar{x})\| \leq B\varepsilon^\sigma$  the intermediate point  $\hat{x}$  belongs to  $\Omega$  for all  $\varepsilon$  sufficiently small. Moreover, for all possible moves  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$  such that  $\|\Delta\hat{x} - \varepsilon^{1-\alpha}n(\bar{x})\| \leq B\varepsilon^\sigma$  and  $\Delta x$  determined by (1.2.21), we have*

$$\|\Delta\hat{x} - \Delta x\| \geq \varepsilon^{1-\alpha} - \text{dist}(x) - B\varepsilon^\sigma + O(\varepsilon^{2-2\alpha}). \quad (1.4.5)$$

Furthermore, if in addition we assume  $\text{dist}(x) \geq \varepsilon^{1-\alpha} - \varepsilon^\eta$  with  $1 - \alpha < \eta < \sigma$ , the intermediate point  $\hat{x}$  is outside  $\Omega$  for all  $\varepsilon$  sufficiently small.

*Proof.* For the first assertion, since  $\Omega$  satisfies the uniform interior ball condition (there is no neck pitching for  $\varepsilon$  sufficiently small), we observe that the set  $\partial\Omega \cap B(x, 2\varepsilon^{1-\alpha})$  is below a paraboloid  $P_1$  of opening  $A$  and above a paraboloid  $P_2$  of opening  $-A$  touching  $\partial\Omega$  at  $\bar{x}$ . By the Taylor expansion, if  $T_{\bar{x}}\partial\Omega$  denotes the tangent space to  $\partial\Omega$  at  $\bar{x}$ , we get that for all  $y \in \partial\Omega \cap B(x, 2\varepsilon^{1-\alpha})$ ,

$$|(y - \bar{x}) \cdot n(\bar{x})| = \text{dist}(y, T_{\bar{x}}\partial\Omega) \leq \frac{1}{2}A(2\varepsilon^{1-\alpha})^2,$$

Since  $(x + \Delta\hat{x} - \bar{x}) \cdot n(\bar{x}) \leq -\varepsilon^{1-\alpha} - \text{dist}(x) + B\varepsilon^\sigma$ , we deduce that for all  $\varepsilon$  sufficiently small,

$$(x + \Delta\hat{x} - \bar{x}) \cdot n(\bar{x}) < \inf_{y \in \partial\Omega \cap B(x, 2\varepsilon^{1-\alpha})} (y - \bar{x}) \cdot n(\bar{x}),$$

which yields that  $x + \Delta\hat{x}$  belongs to  $\Omega$ .

For the second claim, we denote by  $(\kappa_i(x))_{1 \leq i \leq d-1}$  the principal curvatures at  $x$  on  $\partial\Omega$  and by  $(e_1, \dots, e_d)$  an orthonormal frame centered in  $\bar{x}$  with first vector  $e_1 = n(\bar{x})$ . Since  $\Omega$  is a  $C^2$ -domain,  $(e_2, \dots, e_d)$  form a basis of the tangent space  $T_{\bar{x}}\partial\Omega$ . We compute

$$\varepsilon^{1-\alpha} - B\varepsilon^\sigma \leq \Delta\hat{x} \cdot n(\bar{x}) = (\Delta\hat{x} - \varepsilon^{1-\alpha}n(\bar{x})) \cdot n(\bar{x}) + \varepsilon^{1-\alpha}.$$

Thus  $\hat{x}$  is contained in the half-space  $H_1$  determined by  $(y - \bar{x}) \cdot e_1 \geq \varepsilon^{1-\alpha} - \text{dist}(x) - B\varepsilon^\sigma$  and  $\text{dist}(\hat{x}, T_{\bar{x}}\partial\Omega) \geq \varepsilon^{1-\alpha} - \text{dist}(x) - B\varepsilon^\sigma$ . Moreover, we deduce from (1.4.4) and the triangle inequality

that for each move  $\Delta\hat{x}$  we have  $x + \Delta x \in B(\bar{x}, 2\varepsilon^{1-\alpha})$ . Assume  $x_1 = p(x_2, \dots, x_d)$  is a local  $C^2$ -parametrization of  $\partial\Omega$  around  $x$ . By a Taylor argument and by continuity of the principal curvatures on  $\partial\Omega$ , it follows that, for  $\varepsilon > 0$  small enough,

$$\text{dist}(x + \Delta x, T_{\bar{x}}\partial\Omega) \leq \frac{1}{2}C_1(2\varepsilon^{1-\alpha})^2 = 2C_1\varepsilon^{2-2\alpha}, \quad (1.4.6)$$

where  $C_1 := 2 \max \{|\kappa_i(\bar{x})| : 1 \leq i \leq d-1\}$ . By the triangle inequality, we deduce that

$$\begin{aligned} \|x + \Delta x - \hat{x}\| &\geq \|\text{proj}_{T_{\bar{x}}\partial\Omega}(x + \Delta x) - \hat{x}\| - \|x + \Delta x - \text{proj}_{T_{\bar{x}}\partial\Omega}(x + \Delta x)\| \\ &\geq \text{dist}(\hat{x}, T_{\bar{x}}\partial\Omega) - \text{dist}(x + \Delta x, T_{\bar{x}}\partial\Omega) \\ &\geq \varepsilon^{1-\alpha} - \text{dist}(x) - B\varepsilon^\sigma - 2C_1\varepsilon^{2-2\alpha}. \end{aligned}$$

In particular, if  $\text{dist}(x) \geq \varepsilon^{1-\alpha} - \varepsilon^\eta$  with  $1 - \alpha < \eta < \sigma$  the right-hand side is strictly positive for  $\varepsilon$  sufficiently small and  $\hat{x} \notin \Omega$ .  $\square$

The next lemmas gather some estimates which will be useful to establish our consistency estimates.

**Lemma 1.4.3.** *Under the hypothesis of Lemma 1.4.2, for all moves  $\Delta\hat{x}$  constrained by (1.2.17), determining  $\Delta x$  by (1.2.21), we have*

$$-\frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) \leq -\frac{1}{2}\left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right)(\Delta\hat{x}) \cdot n(\bar{x}) + \|\Delta\hat{x} - \Delta x\| \leq \frac{3}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)). \quad (1.4.7)$$

*Proof.* The left-hand side of (1.4.7) can be written in the form

$$-\frac{1}{2}\left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right)(\Delta\hat{x}) \cdot n(\bar{x}) + \|\Delta\hat{x} - \Delta x\| = (\varepsilon^{1-\alpha} - \text{dist}(x))\left[-\frac{1}{2}\frac{(\Delta\hat{x}) \cdot n(\bar{x})}{\varepsilon^{1-\alpha}} + \frac{\|\Delta\hat{x} - \Delta x\|}{\varepsilon^{1-\alpha} - \text{dist}(x)}\right],$$

which directly gives the desired estimates by using (1.2.17) and the first inequality given by (1.4.4).  $\square$

**Lemma 1.4.4.** *Let  $A \in \mathcal{M}^N(\mathbb{R})$ ,  $k \in C_b(\partial\Omega)$  extended by some function  $k : \bar{\Omega} \rightarrow \mathbb{R}$ , and  $x \in \bar{\Omega}$ . Suppose in addition that*

$$(3\varepsilon^{1-\alpha} - \text{dist}(x))\|A\| \leq \inf_{x + \frac{\Delta\hat{x}}{\Delta\hat{x}} \notin \Omega} k(x + \Delta x), \quad (1.4.8)$$

*with  $\Delta\hat{x}$  constrained by (1.2.17) and  $\Delta x$  determined by (1.2.21). Then*

$$\min_{\Delta\hat{x}} \{\langle A\Delta x, \Delta x \rangle - \langle A\Delta\hat{x}, \Delta\hat{x} \rangle + \|\Delta\hat{x} - \Delta x\|k(x + \Delta x)\} = 0, \quad (1.4.9)$$

*where  $\Delta\hat{x}$  is constrained by (1.2.17) and determines  $\Delta x$  by (1.2.21).*

*Proof.* If  $\hat{x} = x + \Delta\hat{x} \in \bar{\Omega}$ , the function is equal to zero. We now consider the moves for which  $\hat{x} \notin \bar{\Omega}$ . Then

$$\langle A\Delta x, \Delta x \rangle - \langle A\Delta\hat{x}, \Delta\hat{x} \rangle = \langle A(\Delta\hat{x} - \Delta x), \Delta\hat{x} - \Delta x \rangle + 2\langle A\Delta\hat{x}, \Delta\hat{x} - \Delta x \rangle. \quad (1.4.10)$$

By the Cauchy-Schwarz inequality, we obtain

$$|\langle A\Delta x, \Delta x \rangle - \langle A\Delta\hat{x}, \Delta\hat{x} \rangle| \leq \|A\| \|\Delta\hat{x} - \Delta x\| (\|\Delta\hat{x} - \Delta x\| + 2\|\Delta\hat{x}\|). \quad (1.4.11)$$

By using (1.4.4) and  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , we get

$$|\langle A\Delta x, \Delta x \rangle - \langle A\Delta\hat{x}, \Delta\hat{x} \rangle| \leq \|A\| \|\Delta\hat{x} - \Delta x\| (3\varepsilon^{1-\alpha} - \text{dist}(x)). \quad (1.4.12)$$

Thus

$$\begin{aligned} &\langle A\Delta x, \Delta x \rangle - \langle A\Delta\hat{x}, \Delta\hat{x} \rangle + \|\Delta\hat{x} - \Delta x\|k(x + \Delta x) \\ &\geq \|\Delta\hat{x} - \Delta x\| \left\{ \inf_{x + \frac{\Delta\hat{x}}{\Delta\hat{x}} \notin \Omega} k(x + \Delta x) - \|A\|(3\varepsilon^{1-\alpha} - \text{dist}(x)) \right\}. \end{aligned}$$

The right-hand side of this last inequality is strictly positive by the assumption (1.4.8).  $\square$

### 1.4.1.2 Consistency estimates

In this subsection we state our consistency estimates. They explain precisely the conditions under which the usual estimate proposed in [88] holds for  $x$  near the boundary and  $\phi \in C^2(\overline{\Omega})$ . If it does not hold, there is a degeneration of the estimates respecting the final discussion of formal derivation of the PDE at Section 1.2.2.1. For fixed  $x \in \Omega(\varepsilon^{1-\alpha})$ , these estimates take into account the size and the sign of the boundary condition in the small ball  $B(x, \varepsilon^{1-\alpha})$  and the distance  $\text{dist}(x)$  to the boundary.

In the heuristic derivation presented in Section 1.2.2.1, we assumed that the map

$$\Delta \hat{x} \mapsto h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x),$$

with  $\Delta x$  determined by (1.2.21), was locally constant in a  $\delta$ -neighborhood of the boundary near  $x$ . In the general case, this hypothesis must be relaxed. To do this, we observe that, for all  $\Delta \hat{x}$  constrained by (1.2.17) satisfying  $x + \Delta \hat{x} \notin \Omega$  and determining  $\Delta x$  by (1.2.21),

$$m_\varepsilon^x[\phi] \leq h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x) \leq M_\varepsilon^x[\phi],$$

where  $m_\varepsilon^x[\phi]$  and  $M_\varepsilon^x[\phi]$  are defined by (1.3.1)–(1.3.2). Therefore we are going to specify some strategies for Helen which are associated to the two extreme situations  $m_\varepsilon^x[\phi]$  and  $M_\varepsilon^x[\phi]$  by following the optimal choices (1.2.48) and (1.2.52) obtained in the formal derivation at Section 1.2.2.1. More precisely, for all  $x \in \Omega(\varepsilon^{1-\alpha})$ , we define the strategies  $p_{\text{opt}}^m(x)$ ,  $p_{\text{opt}}^M(x)$  and  $\Gamma_{\text{opt}}(x)$  in an orthonormal basis  $\mathcal{B} = (e_1 = n(\bar{x}), e_2, \dots, e_d)$  respectively by

$$p_{\text{opt}}^m(x) := D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) m_\varepsilon^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] n(\bar{x}), \quad (1.4.13)$$

$$p_{\text{opt}}^M(x) := D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) M_\varepsilon^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] n(\bar{x}), \quad (1.4.14)$$

and

$$\Gamma_{\text{opt}}(x) := D^2\phi(x) + \left[ \frac{1}{2} \left( -1 + \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] E_{11}, \quad (1.4.15)$$

where  $E_{11}$  denotes the unit-matrix  $(1, 1)$  in the basis  $\mathcal{B}$ . These strategies depend on the local behavior of  $\phi$  (size and amplitude) around the boundary and on the geometry of the boundary itself.

Since there is a degeneration of the usual estimates, there is no hope for one simple estimate. We are going to separate the study in two steps: Proposition 1.4.5 provides the estimates for the lower bound and Proposition 1.4.10 deals with the upper bound. Moreover, Section 1.4.2 is devoted to the technical proof of the upper bound distinguishing several cases according to the size of  $M_\varepsilon^x[\phi]$  and  $\text{dist}(x)$ .

**Proposition 1.4.5.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15). Let  $p_{\text{opt}}^m$  and  $\Gamma_{\text{opt}}$  be respectively defined in the orthonormal basis  $(e_1 = n(\bar{x}), e_2, \dots, e_d)$  by (1.4.13) and (1.4.15). For any  $x, t, z$  and any smooth function  $\phi$  defined near  $x$ ,  $S_\varepsilon[x, t, z, \phi]$  being defined by (1.4.1), we distinguish two cases:*

*i. Big bonus: if  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$  or  $m_\varepsilon^x[\phi] > \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , then*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \geq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)).$$

*ii. Penalty or small bonus: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $m_\varepsilon^x[\phi] \leq \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , then*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \geq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) (sm_\varepsilon^x[\phi] - 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}) - \varepsilon^2 \mathcal{F}(t, x, z, p_{\text{opt}}^m(x), \Gamma_{\text{opt}}(x)),$$

*where  $s = -1$  if  $m_\varepsilon^x[\phi] \geq 0$  and  $s = 3$  if  $m_\varepsilon^x[\phi] < 0$ .*

*Proof.* If  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ , the usual estimate [88, Lemma 4.1] holds. We now focus on the case  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$ . By the definition of  $m_\varepsilon^x[\phi]$  given by (1.3.1) and the positivity of  $\|\Delta\hat{x} - \Delta x\|$ , for all  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , we have

$$\|\Delta\hat{x} - \Delta x\| \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\} \geq \|\Delta\hat{x} - \Delta x\| m_\varepsilon^x[\phi]. \quad (1.4.16)$$

Therefore it is sufficient to find a lower bound for

$$\max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (D\phi(x) - p) \cdot \Delta\hat{x} + m_\varepsilon^x[\phi] \|\Delta\hat{x} - \Delta x\| + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right].$$

where  $p$ ,  $\Gamma$  and  $\Delta\hat{x}$  are constrained by (1.2.16)–(1.2.17) and  $\Delta x$  determined by (1.2.21). In other words, by taking advantage of the monotonicity of the operator  $S_\varepsilon$  with (1.4.16), we shall look for a lower bound for an approximated operator bounding  $S_\varepsilon$  from below and very close to it when  $\varepsilon \rightarrow 0$ .

Then, we also observe that for every choice  $p$  and  $\Gamma$ ,

$$\begin{aligned} S_\varepsilon[x, t, z, \phi] - \phi(x) &\geq -\varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ &\quad + \min_{\Delta\hat{x}} \left[ (D\phi(x) - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle + \|\Delta\hat{x} - \Delta x\| m_\varepsilon^x[\phi] \right]. \end{aligned}$$

We now distinguish two particular strategies for Helen. For part (i), we consider the particular choice  $p = D\phi(x)$ ,  $\Gamma = D^2\phi(x)$  and obtain

$$\begin{aligned} S_\varepsilon[x, t, z, \phi] - \phi(x) &\geq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)) \\ &\quad + \min_{\Delta\hat{x}} \left[ \frac{1}{2} (\langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle D^2\phi(x) \Delta\hat{x}, \Delta\hat{x} \rangle + \|\Delta\hat{x} - \Delta x\| m_\varepsilon^x[\phi]) \right] \\ &\geq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)), \end{aligned}$$

by applying Lemma 1.4.4 with  $A = \frac{1}{2} D^2\phi(x)$ . For part (ii), we consider the choice  $p = p_{\text{opt}}^m(x)$ ,  $\Gamma = \Gamma_{\text{opt}}(x)$  and find

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \geq -\varepsilon^2 \mathcal{F}(t, x, z, p_{\text{opt}}^m(x), \Gamma_{\text{opt}}(x)) + l^x[\phi],$$

with  $l^x[\phi]$  defined by

$$l^x[\phi] := \min_{\Delta\hat{x}} \left[ (D\phi(x) - p_{\text{opt}}^m) \cdot (\Delta\hat{x}) + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma_{\text{opt}}(x) \Delta\hat{x}, \Delta\hat{x} \rangle + \|\Delta\hat{x} - \Delta x\| m_\varepsilon^x[\phi] \right]. \quad (1.4.17)$$

It now remains to give a lower bound for  $l^x[\phi]$ . By plugging the expression (1.4.13) of  $p_{\text{opt}}^m(x)$  in (1.4.17), we have

$$\begin{aligned} l^x[\phi] &= \min_{\Delta\hat{x}} \left[ \left( -\frac{\varepsilon^{1-\alpha} - \text{dist}(x)}{2\varepsilon^{1-\alpha}} (\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| \right) m_\varepsilon^x[\phi] \right. \\ &\quad \left. + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma_{\text{opt}}(x) \Delta\hat{x}, \Delta\hat{x} \rangle + \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2\phi(x))_{11} (\Delta\hat{x})_1 \right]. \end{aligned}$$

It is clear that  $l^x[\phi] \geq l_1^x[\phi] + l_2^x[\phi]$  with  $l_1^x[\phi]$  and  $l_2^x[\phi]$  respectively defined by

$$l_1^x[\phi] := \min_{\Delta\hat{x}} \left[ \left( -\frac{\varepsilon^{1-\alpha} - \text{dist}(x)}{2\varepsilon^{1-\alpha}} (\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| \right) m_\varepsilon^x[\phi] \right], \quad (1.4.18)$$

and

$$l_2^x[\phi] := \frac{1}{2} \min_{\Delta\hat{x}} \left[ \langle D^2\phi(x) \Delta x, \Delta x \rangle - \langle \Gamma_{\text{opt}}(x) \Delta\hat{x}, \Delta\hat{x} \rangle + \frac{\varepsilon^{1-\alpha}}{2} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} (\Delta\hat{x})_1 \right]. \quad (1.4.19)$$



By using Lemmas 1.4.6 and 1.4.8 stated below, giving lower bounds respectively for  $l_1^x[\phi]$  and  $l_2^x[\phi]$ , one obtains

$$\begin{aligned} l^x[\phi] &\geq \frac{s}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m_\varepsilon^x[\phi] - 2\|D^2\phi(x)\|\varepsilon^{1-\alpha}(\varepsilon^{1-\alpha} - \text{dist}(x)) \\ &\geq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))(sm_\varepsilon^x[\phi] - 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}), \end{aligned}$$

which gives the desired estimate.  $\square$

The three following lemmas provide the required estimates for  $l_1^x[\phi]$  and  $l_2^x[\phi]$ .

**Lemma 1.4.6.** *For any  $x \in \Omega(\varepsilon^{1-\alpha})$  and any function  $\phi$  defined at  $x$ ,  $l_1^x[\phi]$  being defined by (1.4.18), we have*

$$\frac{s}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m_\varepsilon^x[\phi] \leq l_1^x[\phi] \leq 0,$$

with  $s = -1$  if  $m_\varepsilon^x[\phi]$  is positive and  $s = 3$  if  $m_\varepsilon^x[\phi]$  is nonpositive.

*Proof.* By considering  $\Delta\hat{x} = 0$ ,  $l_1^x[\phi]$  is negative. To find a lower bound on  $l_1^x[\phi]$ , if  $m_\varepsilon^x[\phi]$  is negative, we may write

$$\left[ -\frac{\varepsilon^{1-\alpha} - \text{dist}(x)}{2\varepsilon^{1-\alpha}}(\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| \right] m_\varepsilon^x[\phi] \geq \frac{3}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))m_\varepsilon^x[\phi],$$

the last inequality being provided by the right-hand side inequality given in Lemma 1.4.3 since by hypothesis  $m_\varepsilon^x[\phi]$  is negative. If  $m_\varepsilon^x[\phi]$  is nonnegative, the result follows from applying the left-hand side inequality given in Lemma 1.4.3.  $\square$

**Lemma 1.4.7.** *Let  $x \in \Omega(\varepsilon^{1-\alpha})$  and  $\phi \in C^2(\overline{\Omega})$ . For all  $\Delta\hat{x}$  constrained by (1.2.17), we have*

$$\left| \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle D^2\phi(x)\Delta\hat{x}, \Delta\hat{x} \rangle \right| \leq \frac{1}{2}\|D^2\phi(x)\| (3\varepsilon^{1-\alpha} - \text{dist}(x)) \|\Delta\hat{x} - \Delta x\|, \quad (1.4.20)$$

and

$$\left| \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma_{\text{opt}}(x)\Delta\hat{x}, \Delta\hat{x} \rangle \right| \leq \frac{1}{4}\|D^2\phi(x)\| (\varepsilon^{1-\alpha} - \text{dist}(x)) (7\varepsilon^{1-\alpha} - \text{dist}(x)), \quad (1.4.21)$$

where  $\Gamma_{\text{opt}}(x)$  is the optimal choice defined by (1.4.15) in an orthonormal basis  $\mathcal{B} = (e_1 = n(\bar{x}), \dots, e_d)$ .

*Proof.* The first inequality is an immediate consequence of (1.4.11). For the second inequality, all the coordinates  $\langle (D^2\phi(x) - \Gamma_{\text{opt}}(x))e_i, e_j \rangle$  in the basis  $\mathcal{B}$  are equal to zero, except for  $i = j = 1$ . By using the vector decomposition given by (1.4.10), we have

$$\begin{aligned} \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma_{\text{opt}}(x)\Delta\hat{x}, \Delta\hat{x} \rangle &= \frac{1}{2}(D^2\phi(x) - \Gamma_{\text{opt}}(x))_{11} |(\Delta\hat{x})_1|^2 \\ &\quad + \frac{1}{2}\|\Delta\hat{x} - \Delta x\|^2 \langle (D^2\phi(x)n(x + \Delta x), n(x + \Delta x)) \rangle - \|\Delta\hat{x} - \Delta x\| \langle D^2\phi(x)n(x + \Delta x), \Delta\hat{x} \rangle. \end{aligned}$$

Since  $(D^2\phi(x) - \Gamma_{\text{opt}}(x))_{11} = \frac{1}{2} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11}$  by (1.4.15), one obtains

$$\begin{aligned} \left| \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma_{\text{opt}}(x)\Delta\hat{x}, \Delta\hat{x} \rangle \right| &\leq \|D^2\phi(x)\| \left\{ \frac{1}{4} \left( 1 - \frac{\text{dist}^2(x)}{\varepsilon^{2-2\alpha}} \right) |(\Delta\hat{x})_1|^2 + \frac{1}{2}\|\Delta\hat{x} - \Delta x\|^2 + \|\Delta\hat{x} - \Delta x\| \|\Delta\hat{x}\| \right\}. \end{aligned}$$

The estimate (1.4.21) now follows from (1.2.17) and (1.4.4).  $\square$

**Lemma 1.4.8.** *For any  $x \in \Omega(\varepsilon^{1-\alpha})$  and any function  $\phi$  defined at  $x$ ,  $l_2^x[\phi]$  being defined by (1.4.19), we have*

$$-2\|D^2\phi(x)\|\varepsilon^{1-\alpha}(\varepsilon^{1-\alpha} - \text{dist}(x)) \leq l_2^x[\phi] \leq 0.$$

*Proof.* By considering  $\Delta\hat{x} = 0$ ,  $l_2$  is negative. To obtain a lower bound on  $l_2$ , Lemma 1.4.7, along with the observation  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , gives

$$\begin{aligned} & \frac{1}{2} \left| \langle D^2\phi(x)\Delta x, \Delta x \rangle - \langle \Gamma_{\text{opt}}(x)\Delta\hat{x}, \Delta\hat{x} \rangle + \frac{1}{2} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2\phi(x))_{11} (\Delta\hat{x})_1 \right| \\ & \leq \frac{1}{4} \|D^2\phi(x)\| (\varepsilon^{1-\alpha} - \text{dist}(x)) (7\varepsilon^{1-\alpha} - \text{dist}(x)) + \frac{1}{4} \|D^2\phi(x)\| \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) \varepsilon^{1-\alpha} \\ & \leq 2 \|D^2\phi(x)\| (\varepsilon^{1-\alpha} - \text{dist}(x)) \varepsilon^{1-\alpha}, \end{aligned} \quad (1.4.22)$$

which is precisely the proposed estimate.  $\square$

We shall now provide the consistency estimates about the upper bound of (1.4.3). Before stating our main estimate in Proposition 1.4.10, we can give a simple case for which the usual estimate holds.

**Lemma 1.4.9.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15). For any  $x, t, z$  and any smooth function  $\phi$  defined near  $x$ ,  $S_\varepsilon[x, t, z, \phi]$  being defined by (1.4.1), if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $M_\varepsilon^x[\phi] \leq -\frac{1}{2} \|D^2\phi(x)\| (3\varepsilon^{1-\alpha} - \text{dist}(x))$ , then we have*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)) + o(\varepsilon^2).$$

Moreover, the implicit constant in the error term is uniform as  $x, t$  and  $z$  range over a compact subset of  $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ .

In the rest of the section, we now focus on the case  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$ . The goal is to obtain precise estimates on (1.4.3) in the following three cases:  $M_\varepsilon^x[\phi]$  very negative,  $M_\varepsilon^x[\phi]$  very positive and  $M_\varepsilon^x[\phi]$  close to zero, the bounds between the cases depending on some powers of  $\varepsilon$ . We have formally shown in Section 1.2.2.1 that the first case is favorable to Mark since Helen can undergo a big penalty if Mark chooses to cross the boundary. On the contrary, the second case is preferable to Helen because she can receive a big coupon if the boundary is crossed. In the last case, the boundary is transparent (think of  $M_\varepsilon^x[\phi] = 0$ ) and the penalization due to the boundary is to be considered only through second order terms. In order to establish accurate upper bounds on (1.4.3), we successively introduce two additional parameters  $\rho, \kappa > 0$  such that

$$1 - \alpha < \rho < \min \left( 1 - \frac{\gamma(r-1)}{2}, 2 - 2\alpha - \gamma \right), \quad (1.4.23)$$

and

$$\gamma + \rho - (1 - \alpha) < \kappa < 1 - \alpha. \quad (1.4.24)$$

These coefficients are well-defined by virtue of (1.2.13) and (1.2.15).

**Proposition 1.4.10.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma, \rho, \kappa$  satisfy (1.2.13)–(1.2.15) and (1.4.23)–(1.4.24). Let  $p_{\text{opt}}^M$  and  $\Gamma_{\text{opt}}$  be respectively defined in the orthonormal basis  $(e_1 = n(\bar{x}), e_2, \dots, e_d)$  by (1.4.14) and (1.4.15). For any  $x, t, z$  and any smooth function  $\phi$  defined near  $x$ ,  $S_\varepsilon[x, t, z, \phi]$  being defined by (1.4.1), we distinguish four cases:*

i. *Big bonus: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $M_\varepsilon^x[\phi] > \frac{4}{3} \|D^2\phi(x)\| \varepsilon^{1-\alpha}$ , then*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq 3(\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi] - \varepsilon^2 \mathcal{F}(t, x, z, p_{\text{opt}}^M(x), \Gamma_{\text{opt}}(x)) + o(\varepsilon^2).$$

ii. *Far from the boundary with a small bonus: if  $\varepsilon^{1-\alpha} - \varepsilon^\rho \leq \text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $M_\varepsilon^x[\phi] \leq \frac{4}{3} \|D^2\phi(x)\| \varepsilon^{1-\alpha}$ , or if  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ , then*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)) + o(\varepsilon^2).$$

- iii. Close to the boundary with a small bonus/penalty: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  and  $-\varepsilon^{1-\alpha-\kappa} \leq M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , then

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x) + C_1 I) + o(\varepsilon^2),$$

$$\text{with } C_1 = \frac{20}{3}\|D^2\phi(x)\| \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right).$$

- iv. Close to the boundary with a big penalty: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  and  $M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}$ , then

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2 \min_{p \in B(p_{\text{opt}}^M(x), r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}(x)) + o(\varepsilon^2), \quad (1.4.25)$$

$$\text{with } r \text{ defined by } r = 3\left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right)|M_\varepsilon^x[\phi]|.$$

Moreover, the implicit constants in the error term is uniform as  $x, t$  and  $z$  range over a compact subset of  $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ .

Before proving these estimates, it is worth drawing a parallel with the formal derivation done at Section 1.2.2.1. The lower bound proposed by Proposition 1.4.5 case (i) corresponds to the formal analysis when  $m > 0$ . The upper bound proposed by Proposition 1.4.10 case (iv) is associated to the formal analysis when  $m < 0$ . Furthermore, we can observe in the proof that the factor  $1/4$  in (1.4.25) could be replaced by any number in  $[1/4, 1/2)$ , the bound  $1/2$  corresponding to the heuristic derivation given by (1.2.53).

## 1.4.2 Proof of Lemma 1.4.9 and Proposition 1.4.10

For sake of notational simplicity, we write  $\lambda_{\min}(A)$  for the smallest eigenvalue of the symmetric matrix  $A$  and we omit the  $x$ -dependence of  $p_{\text{opt}}^M(x)$  and  $\Gamma_{\text{opt}}(x)$ . Moreover, by the definition of  $M_\varepsilon^x[\phi]$  given by (1.3.2) and the positivity of  $\|\Delta\hat{x} - \Delta x\|$ , we get that for all  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ ,

$$\|\Delta\hat{x} - \Delta x\| \{h(x + \Delta x) - D\phi(x) \cdot n(x + \Delta x)\} \leq \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi]. \quad (1.4.26)$$

Therefore it is sufficient to find an upper bound for

$$\max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (D\phi(x) - p) \cdot \Delta\hat{x} + M_\varepsilon^x[\phi] \|\Delta\hat{x} - \Delta x\| + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right].$$

In other words, by taking advantage of the monotonicity of the operator  $S_\varepsilon$  with (1.4.26), we shall look for an upper bound for an approximated operator bounding  $S_\varepsilon$  above and very close to it as  $\varepsilon \rightarrow 0$ .

### 1.4.2.1 Proof of Lemma 1.4.9

We introduce

$$\begin{aligned} \mathcal{A}^x(p, \Gamma, \Delta\hat{x}) &:= (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] \\ &\quad + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma), \end{aligned} \quad (1.4.27)$$

where  $\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x}) - x$ . We give the following useful decomposition:

$$\begin{aligned} \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle \\ = \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle D^2\phi(x) \Delta\hat{x}, \Delta\hat{x} \rangle + \frac{1}{2} \langle (D^2\phi(x) - \Gamma) \Delta\hat{x}, \Delta\hat{x} \rangle, \end{aligned} \quad (1.4.28)$$

which will be used repeatedly in this section. By (1.4.20) we have

$$\begin{aligned} \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2} \langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2} \langle D^2\phi(x)\Delta\hat{x}, \Delta\hat{x} \rangle \\ \leq \|\Delta\hat{x} - \Delta x\| \left( M_\varepsilon^x[\phi] + \frac{1}{2} \|D^2\phi(x)\| (3\varepsilon^{1-\alpha} - \text{dist}(x)) \right) \leq 0. \end{aligned}$$

From the previous inequality and (1.4.28) we deduce that for all  $p, \Gamma, \Delta\hat{x}$  constrained by (1.2.16)–(1.2.17),

$$\mathcal{A}^x(p, \Gamma, \Delta\hat{x}) \leq (D\phi(x) - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (D^2\phi(x) - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma).$$

By monotonicity of the operator  $S_\varepsilon$  and by using [88, Lemma 4.1] to estimate the max min, we conclude that

$$\begin{aligned} S_\varepsilon[x, t, z, \phi] - \phi(x) &\leq \max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (D\phi(x) - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (D^2\phi(x) - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right] \\ &\leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)) + o(\varepsilon^2), \end{aligned}$$

which gives the desired result.

#### 1.4.2.2 Proof of Proposition 1.4.10 case (i)

Define the function  $\mathcal{A}_b^x$  of  $\Delta\hat{x}$  associated to the particular choice  $p = p_{\text{opt}}^M$  and  $\Gamma = \Gamma_{\text{opt}}$  by

$$\mathcal{A}_b^x(\Delta\hat{x}) := (D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2} \langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma_{\text{opt}}\Delta\hat{x}, \Delta\hat{x} \rangle, \quad (1.4.29)$$

where  $\Delta x = \text{proj}_{\overline{\Omega}}(x + \Delta\hat{x}) - x$ . Thus, the operator  $S_\varepsilon$  can be written in the form

$$S_\varepsilon[x, t, z, \phi] - \phi(x) = \max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ \mathcal{A}_b^x(\Delta\hat{x}) + (p_{\text{opt}}^M - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (\Gamma_{\text{opt}} - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right]. \quad (1.4.30)$$

To compute an upper bound of (1.4.30), we now introduce two preliminary lemmas.

**Lemma 1.4.11.** *Assume that  $M_\varepsilon^x[\phi] \geq 0$ . Then  $\mathcal{A}_b^x$  defined by (1.4.29) is  $\Delta\hat{x}$ -bounded by*

$$0 \leq \sup_{\Delta\hat{x}} \mathcal{A}_b^x(\Delta\hat{x}) \leq \frac{1}{2} (\varepsilon^{1-\alpha} - \text{dist}(x)) (3M_\varepsilon^x[\phi] + 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}), \quad (1.4.31)$$

where  $\Delta\hat{x}$  is constrained by (1.2.17).

*Proof.* This estimate follows exactly the same lines as for Lemmas 1.4.6–1.4.8. The sup is clearly positive by considering  $\Delta\hat{x} = 0$ . Then, by plugging the expression of  $p_{\text{opt}}^M$  in  $\mathcal{A}_b^x(\Delta\hat{x})$ , we have

$$\begin{aligned} \mathcal{A}_b^x(\Delta\hat{x}) &= \left\{ -\frac{\varepsilon^{1-\alpha} - \text{dist}(x)}{2\varepsilon^{1-\alpha}} (\Delta\hat{x})_1 + \|\Delta\hat{x} - \Delta x\| \right\} M_\varepsilon^x[\phi] \\ &\quad + \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2\phi(x))_{11} (\Delta\hat{x})_1 + \frac{1}{2} \langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma_{\text{opt}}\Delta\hat{x}, \Delta\hat{x} \rangle. \end{aligned}$$

Since  $M_\varepsilon^x[\phi] \geq 0$ , the estimates (1.4.7) and (1.4.22) give the desired estimate.  $\square$

**Lemma 1.4.12.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15). Let  $(p_\varepsilon)_{0 < \varepsilon \leq 1}$  and  $(\Gamma_\varepsilon)_{0 < \varepsilon \leq 1}$  be two sequences bounded respectively in  $\mathbb{R}^d$  and  $\mathcal{S}_d$ . Then for any  $x, t$  and  $z$ , we have*

$$\max_{\|p\| \leq \varepsilon^{-\beta}} \min_{\substack{\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha} \\ \|\Gamma\| \leq \varepsilon^{-\gamma}}} \left[ (p_\varepsilon - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (\Gamma_\varepsilon - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right] = -\varepsilon^2 \mathcal{F}(t, x, z, p_\varepsilon, \Gamma_\varepsilon) + o(\varepsilon^2).$$

Moreover, the implicit constant in the error term is uniform as  $x, t$ , and  $z$  range over a compact subset of  $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ .

*Proof.* It is a direct adaptation of [88, Lemma 4.1] by distinguishing three cases according to the size of  $\|p_\varepsilon - p\|$  and  $\lambda_{\min}(\Gamma_\varepsilon - \Gamma)$ .  $\square$

We can now provide an upper bound on (1.4.30). By Lemma 1.4.11,  $\mathcal{A}_b$  is upper bounded independently of all possible moves  $\Delta\hat{x}$ . It follows from (1.4.30) that

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq \sup_{\Delta\hat{x}} \mathcal{A}_b^x(\Delta\hat{x}) + \max_{p, \Gamma} \min_{\Delta\hat{x}} \left[ (p_{\text{opt}}^M - p) \cdot \Delta\hat{x} - \frac{1}{2} \langle \Gamma_{\text{opt}} \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right].$$

The consistency Lemma 1.4.12 provides an estimate of the max min and one obtains

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq \sup_{\Delta\hat{x}} \mathcal{A}_b^x(\Delta\hat{x}) - \varepsilon^2 \mathcal{F}(t, x, z, p_{\text{opt}}^M, \Gamma_{\text{opt}}) + o(\varepsilon^2).$$

By plugging the upper bound in (1.4.31) of  $\mathcal{A}_b^x$  in the previous inequality, we obtained the desired result.

### 1.4.2.3 Proof of Proposition 1.4.10 case (ii)

It is sufficient to show that for any  $\|p\| \leq \varepsilon^{-\beta}$  and  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ , there exists  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , determining  $\Delta x$  by (1.2.21), such that

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + M_\varepsilon^x[\phi] \|\Delta\hat{x} - \Delta x\| + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle \\ - \varepsilon^2 f(t, x, z, p, \Gamma) \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)) + o(\varepsilon^2), \end{aligned} \quad (1.4.32)$$

with an error estimate  $o(\varepsilon^2)$  that is independent of  $p$  and  $\Gamma$  and locally uniform in  $x, t, z$ . In view of the conditions (1.2.15) and (1.4.23), we can pick  $\mu > 0$  and  $\delta > 0$  such that

$$\mu + \gamma < 1 - \alpha \quad \text{and} \quad \mu + \gamma r < 1 + \alpha, \quad (1.4.33)$$

$$\delta < \min\{2\alpha, \rho - (1 - \alpha)\}. \quad (1.4.34)$$

Now we consider separately the following three cases:

1.  $\|D\phi(x) - p\| \leq \varepsilon^\mu$  and  $\lambda_{\min}(D^2\phi(x) - \Gamma) \geq -\varepsilon^\delta$ ,
2.  $\|D\phi(x) - p\| \leq \varepsilon^\mu$  and  $\lambda_{\min}(D^2\phi(x) - \Gamma) \leq -\varepsilon^\delta$ ,
3.  $\|D\phi(x) - p\| \geq \varepsilon^\mu$ .

For case 1, we choose  $\Delta\hat{x} = 0$ . By a reasoning similar to case 1 in the proof of [88, Lemma 4.1], we obtained the inequality given by (1.4.32).

For cases 2 and 3, in order to use the decomposition (1.4.28), we now give a preliminary inequality. By the inequality (1.4.20) in Lemma 1.4.7, we have

$$\left| \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle D^2\phi(x) \Delta\hat{x}, \Delta\hat{x} \rangle \right| \leq \frac{3}{2} \|D^2\phi(x)\| \|\Delta\hat{x} - \Delta x\| \varepsilon^{1-\alpha},$$

which yields with the assumption  $M_\varepsilon^x[\phi] \leq \frac{4}{3} \|D^2\phi(x)\| \varepsilon^{1-\alpha}$  that

$$M_\varepsilon^x[\phi] \|\Delta\hat{x} - \Delta x\| + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle D^2\phi(x) \Delta\hat{x}, \Delta\hat{x} \rangle \leq \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha} \|\Delta\hat{x} - \Delta x\|. \quad (1.4.35)$$

By combining the geometric estimate (1.4.4) with the assumption  $\text{dist}(x) \geq \varepsilon^{1-\alpha} - \varepsilon^\rho$ , we get that the left-hand side of (1.4.35) is upper bounded by  $\frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho}$ . By using the decomposition (1.4.28), we deduce that it is sufficient to show that there exists  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$  such that

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (D^2\phi(x) - \Gamma) \Delta\hat{x}, \Delta\hat{x} \rangle + \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)). \end{aligned}$$

For case 2, we choose  $\Delta\hat{x}$  to be an eigenvector for the minimum eigenvalue  $\lambda = \lambda_{\min}(D^2\phi(x) - \Gamma)$  of norm  $\varepsilon^{1-\alpha}$ . Notice that since  $\mathcal{F}$  is monotone in its last input, we have

$$\mathcal{F}(t, x, z, p, \Gamma) \geq \mathcal{F}(t, x, z, D^2\phi(x) - \lambda I).$$

Choosing  $\Delta\hat{x}$  as announced, and changing the sign if necessary to make  $(D\phi(x) - p) \cdot \Delta\hat{x} \leq 0$ , we deduce that

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \frac{1}{2} \langle (D^2\phi(x) - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle + \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ \leq \frac{1}{2} \varepsilon^{2-2\alpha} \lambda + \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} - \varepsilon^2 \mathcal{F}(t, x, z, p, D^2\phi(x) - \lambda I). \end{aligned}$$

If  $-1 \leq \lambda \leq -\varepsilon^\delta$  then  $\varepsilon^{2-2\alpha} \lambda \leq -\varepsilon^{2-2\alpha+\delta}$  and  $\mathcal{F}(t, x, z, p, D^2\phi(x) - \lambda I)$  is bounded. Since  $\varepsilon^{1-\alpha+\rho} \ll \varepsilon^{2-2\alpha+\delta}$  by (1.4.34), for such  $\lambda$  we have

$$\frac{1}{2} \varepsilon^{2-2\alpha} \lambda + \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} - \varepsilon^2 \mathcal{F}(t, x, z, p, D^2\phi(x) - \lambda I) \leq -\frac{1}{4} \varepsilon^{2-2\alpha+\delta} + O(\varepsilon^2).$$

In this case, we are done by (1.4.34), since the right-hand side is  $\leq \varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x))$  when  $\varepsilon$  is small enough.

To complete case 2, suppose  $\lambda \leq -1$ . Then using the growth hypothesis (1.1.7) and recalling that  $p$  is near  $D\phi(x)$  we have

$$\frac{1}{2} \varepsilon^{2-2\alpha} \lambda - \varepsilon^2 \mathcal{F}(t, x, z, p, D^2\phi(x) - \lambda I) \leq -\frac{1}{2} \varepsilon^{2-2\alpha} |\lambda| + C\varepsilon^2(1 + |\lambda|^r).$$

Now notice that  $|\lambda| \leq C(1 + \|\Gamma\|) \leq C\varepsilon^{-\gamma}$ . Since  $\gamma(r-1) < 2\alpha$  we have  $\varepsilon^{2-2\alpha} |\lambda| \gg \varepsilon^2 |\lambda|^r$ . Therefore we deduce by (1.4.34) that

$$-\frac{1}{2} \varepsilon^{2-2\alpha} |\lambda| + C\varepsilon^2 |\lambda|^r + \frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} \leq -\frac{1}{4} \varepsilon^{2-2\alpha} \leq -\varepsilon^2 \mathcal{F}(t, x, z, D\phi(x), D^2\phi(x)),$$

when  $\varepsilon$  is sufficiently small. Case 2 is now complete.

Finally, to treat case 3, we take  $\Delta\hat{x}$  parallel to  $D\phi(x) - p$  with norm  $\varepsilon^{1-\alpha}$ , and with the sign chosen such that

$$(D\phi(x) - p) \cdot \Delta\hat{x} = -\varepsilon^{1-\alpha} \|D\phi(x) - p\| \leq -\varepsilon^{1-\alpha+\mu}.$$

By observing that  $\frac{17}{6} \|D^2\phi(x)\| \varepsilon^{1-\alpha+\rho} \ll \varepsilon^{1-\alpha} \|D\phi(x) - p\|$ , this case follows exactly the same lines as [88, Lemma 4.1].

#### 1.4.2.4 Proof of Proposition 1.4.10 case (iii)

This proof is quite similar to case (ii). Since this estimate will not be needed in the rest of the paper, we just indicate that we need to distinguish three cases according to the respective sizes of  $\|D\phi(x) - p\|$  and  $\lambda_{\min}(D^2\phi(x) - \Gamma)$  with respect to  $\varepsilon^\mu$  and  $-C_1 - \varepsilon^\alpha$ , where  $\mu$  is defined by (1.4.33).

#### 1.4.2.5 Proof of Proposition 1.4.10 case (iv)

This case corresponds to the heuristic derivation presented at Section 1.2.2.1 when  $m < 0$ . Recalling that  $p_{\text{opt}}^M$  and  $\Gamma_{\text{opt}}$  are defined by (1.4.14)–(1.4.15), our task is to show that for any  $\|p\| \leq \varepsilon^{-\beta}$  and  $33\|\Gamma\| \leq \varepsilon^{-\gamma}$ , there exists  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ , determining  $\Delta x$  by (1.2.21), such that

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2} \langle D^2\phi(x) \Delta x, \Delta x \rangle - \frac{1}{2} \langle \Gamma \Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ \leq \frac{1}{4} (\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi] - \varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) + o(\varepsilon^2), \quad (1.4.36) \end{aligned}$$

with an error estimate  $o(\varepsilon^2)$  that is independent of  $p$  and  $\Gamma$  and locally uniform in  $x, t, z$ . We can notice in (1.4.36) that the function  $M_\varepsilon^x[\phi]$  is  $\varepsilon, x$ -bounded by  $\|h\|_{L^\infty} + \|D\phi\|_{L^\infty}$ . Moreover, by Lemma 1.4.7 we have

$$\frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma_{\text{opt}}\Delta\hat{x}, \Delta\hat{x} \rangle \leq \frac{7}{4}\|D^2\phi(x)\|(\varepsilon^{1-\alpha} - \text{dist}(x))\varepsilon^{1-\alpha}. \quad (1.4.37)$$

Thus, it is sufficient to examine, for any  $\|p\| \leq \varepsilon^{-\beta}$  and  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ ,

$$\min_{\Delta\hat{x}} \left[ (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2}\langle (\Gamma_{\text{opt}} - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \right]. \quad (1.4.38)$$

We consider separately the following three cases:

- a.  $\|p_{\text{opt}}^M - p\| \leq 3 \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\phi]|$ , and  $\lambda_{\min}(\Gamma_{\text{opt}} - \Gamma) \geq -\varepsilon^\alpha$ ,
- b.  $\|p_{\text{opt}}^M - p\| \leq 3 \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\phi]|$ , and  $\lambda_{\min}(\Gamma_{\text{opt}} - \Gamma) \leq -\varepsilon^\alpha$ ,
- c.  $\|p_{\text{opt}}^M - p\| \geq 3 \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\phi]|$ .

For case (a), we choose  $\Delta\hat{x} = \pm \varepsilon^{1-\alpha} n(\bar{x})$  with the sign chosen such that

$$(p - p_{\text{opt}}^M) \cdot \Delta\hat{x} \leq 0.$$

Since  $\lambda_{\min}(\Gamma_{\text{opt}} - \Gamma) \geq -\varepsilon^\alpha$  we have  $\Gamma_{\text{opt}} - \Gamma + \varepsilon^\alpha I \geq 0$  and thus  $\Gamma \leq \Gamma_{\text{opt}} + \varepsilon^\alpha I$ . Using the monotonicity of  $\mathcal{F}$  with respect to its last entry, this gives  $\mathcal{F}(t, x, z, p, \Gamma) \geq \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}} + \varepsilon^\alpha I)$ . Since  $\mathcal{F}$  is locally Lipschitz, we conclude that

$$\mathcal{F}(t, x, z, p, \Gamma) \geq \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) + O(\varepsilon^\alpha) \geq \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) + O(\varepsilon^\alpha). \quad (1.4.39)$$

The constant in the error term is independent of  $p$  and  $\Gamma$ , since we are assuming in case (a) that  $\|p - p_{\text{opt}}^M\| \leq 3(\|h\|_{L^\infty} + \|D\phi\|_{L^\infty})$ . Moreover we directly compute

$$(D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] = \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi]. \quad (1.4.40)$$

Since  $\varepsilon^{1-\alpha} - \text{dist}(x) \geq \varepsilon^\rho$  and  $M_\varepsilon^x[\phi] < 0$ , we have

$$\frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi] \leq \frac{1}{2} \varepsilon^\rho M_\varepsilon^x[\phi] \leq -\frac{1}{2} \varepsilon^{1-\alpha-\kappa+\rho}. \quad (1.4.41)$$

By noticing that  $\varepsilon^{2-2\alpha-\gamma} \ll \varepsilon^{1-\alpha-\kappa+\rho}$  using (1.4.24), we deduce from (1.4.41) that

$$\left| \frac{1}{2}\langle (\Gamma_{\text{opt}} - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle \right| \leq \frac{1}{2}(\|D^2\phi(x)\| + \varepsilon^{-\gamma})\varepsilon^{2-2\alpha} \leq \frac{3}{4}\varepsilon^{2-2\alpha-\gamma} \ll (\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi]. \quad (1.4.42)$$

Therefore, by combining (1.4.37), (1.4.40) and (1.4.42), the choice  $\Delta\hat{x} = \pm \varepsilon^{1-\alpha} n(\bar{x})$  in the left-hand side of (1.4.36) yields

$$\begin{aligned} & (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ & \leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) \left( M_\varepsilon^x[\phi] + \frac{7}{2}\|D^2\phi(x)\|\varepsilon^{1-\alpha} \right) + \frac{3}{4}\varepsilon^{2-2\alpha-\gamma} - \varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) + o(\varepsilon^2) \\ & \leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi] - \varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) + o(\varepsilon^2), \end{aligned}$$

as desired.

For case (b), in view of the condition (1.4.23), we can pick  $\sigma > 1 - \alpha$  such that

$$\rho < \sigma < 1 - \frac{\gamma(r-1)}{2}. \quad (1.4.43)$$

Let  $v^\lambda$  be a unit eigenvector for the minimum eigenvalue  $\lambda = \lambda_{\min}(\Gamma_{\text{opt}} - \Gamma)$ . We choose  $\Delta\hat{x}$  of the form

$$\Delta\hat{x} = \pm [(\varepsilon^{1-\alpha} - \varepsilon^\sigma) n(\bar{x}) + \text{sgn}(\langle n(\bar{x}), v^\lambda \rangle) \varepsilon^\sigma v^\lambda] = \pm [a_1 n(\bar{x}) + b v^\lambda], \quad (1.4.44)$$

where  $a_1 = (\varepsilon^{1-\alpha} - \varepsilon^\sigma)$ ,  $b = \text{sgn}(\langle n(\bar{x}), v^\lambda \rangle) \varepsilon^\sigma$  and  $\text{sgn}$  denotes the sign function with the convention that  $\text{sgn}(0) = 1$ . The sign  $\pm$  will be chosen later. This move fulfills the following estimate.

**Lemma 1.4.13.** *The move  $\Delta\hat{x}$  defined by (1.4.44) is authorized by the game and satisfies*

$$(D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] \leq \frac{\varepsilon^{1-\alpha} - \text{dist}(x)}{2} (M_\varepsilon^x[\phi] + \|D^2\phi(x)\| \varepsilon^{1-\alpha} - 4\varepsilon^\sigma M_\varepsilon^x[\phi]), \quad (1.4.45)$$

independently of the choice on  $\pm$  in (1.4.44).

*Proof.* To authorize this move, it suffices to check that  $\|\Delta\hat{x}\| \leq \varepsilon^{1-\alpha}$ . After some calculations and by rearranging the terms, we compute

$$\begin{aligned} \|\Delta\hat{x}\|^2 &= \varepsilon^{2-2\alpha} + 2\varepsilon^{2\sigma} - 2\varepsilon^{1-\alpha+\sigma} + \varepsilon^\sigma (\varepsilon^{1-\alpha} - \varepsilon^\sigma) |\langle n(\bar{x}), v^\lambda \rangle| \\ &= \varepsilon^{2-2\alpha} - 2\varepsilon^{1-\alpha+\sigma} (1 - \varepsilon^{\sigma-1+\alpha}) \left(1 - \frac{1}{2} |\langle n(\bar{x}), v^\lambda \rangle|\right) \leq \varepsilon^{2-2\alpha}. \end{aligned}$$

For the second part, we distinguish successively the two cases  $\pm$ . By (1.4.44), we directly compute

$$\Delta\hat{x} \cdot n(\bar{x}) = \pm [(\varepsilon^{1-\alpha} - \varepsilon^\sigma) + |\langle n(\bar{x}), v^\lambda \rangle| \varepsilon^\sigma] = \pm [\varepsilon^{1-\alpha} - (1 - |\langle n(\bar{x}), v^\lambda \rangle|) \varepsilon^\sigma]. \quad (1.4.46)$$

If  $\Delta\hat{x} \cdot n(\bar{x}) \leq 0$ , this move corresponds to the sign  $-$  in (1.4.44) by (1.4.46) and we observe that  $\hat{x} \in \Omega$  by Lemma 1.4.2. As a result, by introducing the explicit expressions of  $p_{\text{opt}}^M$  and  $(\Delta\hat{x})_1$  respectively given by (1.4.14) and (1.4.46), we get

$$\begin{aligned} (D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] &= (D\phi(x) - p_{\text{opt}}^M)_1 (\Delta\hat{x})_1 \\ &= - \left( -\frac{1}{2} \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) M_\varepsilon^x[\phi] + \frac{1}{4} (\varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}}) (D^2\phi(x))_{11} \right) (\varepsilon^{1-\alpha} - (1 - |\langle n(\bar{x}), v^\lambda \rangle|) \varepsilon^\sigma). \end{aligned} \quad (1.4.47)$$

Since  $0 \leq \varepsilon^{1-\alpha} - (1 - |\langle n(\bar{x}), v^\lambda \rangle|) \varepsilon^\sigma \leq \varepsilon^{1-\alpha}$ , we observe that

$$\begin{aligned} \left| \frac{1}{4} \left( \varepsilon^{1-\alpha} - \frac{\text{dist}^2(x)}{\varepsilon^{1-\alpha}} \right) (D^2\phi(x))_{11} (\varepsilon^{1-\alpha} - (1 - |\langle n(\bar{x}), v^\lambda \rangle|) \varepsilon^\sigma) \right| &\leq \frac{1}{4} \|D^2\phi(x)\| (\varepsilon^{2-2\alpha} - \text{dist}^2(x)) \\ &\leq \frac{1}{2} \|D^2\phi(x)\| (\varepsilon^{1-\alpha} - \text{dist}(x)) \varepsilon^{1-\alpha}. \end{aligned}$$

By plugging this inequality in (1.4.47) and rearranging the terms, we obtain

$$\begin{aligned} (D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] &\leq (\varepsilon^{1-\alpha} - \text{dist}(x)) \left\{ \frac{1}{2} (1 - (1 - |\langle n(\bar{x}), v^\lambda \rangle|) \varepsilon^{\sigma-1+\alpha}) M_\varepsilon^x[\phi] + \frac{1}{2} \|D^2\phi(x)\| \varepsilon^{1-\alpha} \right\} \\ &\leq \frac{1}{2} (\varepsilon^{1-\alpha} - \text{dist}(x)) (M_\varepsilon^x[\phi] + \|D^2\phi(x)\| \varepsilon^{1-\alpha}) - \frac{1}{2} \varepsilon^\sigma M_\varepsilon^x[\phi]. \end{aligned}$$

Otherwise, if  $\Delta\hat{x} \cdot n(\bar{x}) \geq 0$ , this move corresponds to the sign  $+$  in (1.4.44) by (1.4.46). We have

$$\|\Delta\hat{x} - \varepsilon^{1-\alpha} n(\bar{x})\| = \|-\varepsilon^\sigma n(\bar{x}) + \text{sgn}(\langle n(\bar{x}), v^\lambda \rangle) \varepsilon^\sigma v^\lambda\| = \sqrt{2} \varepsilon^\sigma \sqrt{1 - |\langle n(\bar{x}), v^\lambda \rangle|} \leq \sqrt{2} \varepsilon^\sigma.$$

By using Lemma 1.4.2, we deduce from the previous inequality that, for  $\varepsilon$  small enough, the intermediate point  $\hat{x} = x + \Delta\hat{x}$  is outside  $\Omega$  and

$$\varepsilon^{1-\alpha} - \text{dist}(x) - \sqrt{2} \varepsilon^\sigma - 2C_1 \varepsilon^{2-2\alpha} \leq \|\Delta\hat{x} - \Delta x\|, \quad (1.4.48)$$



where  $C_1$  is a certain constant depending on the principal curvatures of  $\partial\Omega$  in a neighborhood of  $x$ . By repeating the computations above, we find

$$\begin{aligned} (D\phi(x) - p_{\text{opt}}^M)_1(\Delta\hat{x})_1 &\leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) \{-(1 - (1 - |\langle n(\bar{x}), v^\lambda \rangle|)\varepsilon^{\sigma-1+\alpha})M_\varepsilon^x[\phi] + \|D^2\phi(x)\|\varepsilon^{1-\alpha}\} \\ &\leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))(-M_\varepsilon^x[\phi] + \|D^2\phi(x)\|\varepsilon^{1-\alpha}). \end{aligned}$$

Recalling that  $M_\varepsilon^x[\phi] < 0$ , by combining (1.4.48) with the previous estimate, we are led to

$$\begin{aligned} (D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\|M_\varepsilon^x[\phi] &\leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))(-M_\varepsilon^x[\phi] + \|D^2\phi(x)\|\varepsilon^{1-\alpha}) + (\varepsilon^{1-\alpha} - \text{dist}(x) - \sqrt{2}\varepsilon^\sigma - 2C_1\varepsilon^{2-2\alpha})M_\varepsilon^x[\phi] \\ &\leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x))(M_\varepsilon^x[\phi] + \|D^2\phi(x)\|\varepsilon^{1-\alpha}) - \varepsilon^\sigma M_\varepsilon^x[\phi](\sqrt{2} + 2C_1\varepsilon^{2-2\alpha-\sigma}). \end{aligned}$$

Putting together the two cases, the proof of the inequality given by (1.4.45) is complete.  $\square$

Now we turn back to the analysis of case (b). Note that since  $\mathcal{F}$  is monotone in its last input

$$\mathcal{F}(t, x, z, p, \Gamma) \geq \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}} - \lambda I).$$

The direct evaluation of the second order terms in  $\Delta\hat{x}$  of (1.4.38) gives

$$\begin{aligned} \langle (\Gamma_{\text{opt}} - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle &= a_1^2 \langle (\Gamma_{\text{opt}} - \Gamma)n(\bar{x}), n(\bar{x}) \rangle + 2a_1b \langle (\Gamma_{\text{opt}} - \Gamma)v^\lambda, n(\bar{x}) \rangle + b^2 \langle (\Gamma_{\text{opt}} - \Gamma)v^\lambda, v^\lambda \rangle \\ &\leq a_1^2(\|\Gamma_{\text{opt}}\| + \|\Gamma\|) + 2a_1b\lambda \langle v^\lambda, n(\bar{x}) \rangle + b^2\lambda. \end{aligned}$$

With our choice for  $\Delta\hat{x}$ , we have  $a_1b \langle v^\lambda, n(\bar{x}) \rangle \geq 0$ . Hence, since  $\lambda \leq 0$  in case (b), it follows that

$$\langle (\Gamma_{\text{opt}} - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle \leq a_1^2(\|\Gamma_{\text{opt}}\| + \|\Gamma\|) + b^2\lambda \leq \varepsilon^{2-2\alpha}(\|D^2\phi(x)\| + \varepsilon^{-\gamma}) + \varepsilon^{2\sigma}\lambda.$$

Choosing  $\Delta\hat{x}$  as announced, using (1.4.37) and (1.4.45) and changing the sign  $\pm$  in (1.4.44) if necessary to make  $(p_{\text{opt}} - p) \cdot \Delta\hat{x} \leq 0$ ,

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\|M_\varepsilon^x[\phi] &+ \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2\mathcal{F}(t, x, z, p, \Gamma) \\ &\leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) \left( M_\varepsilon^x[\phi] + \frac{9}{2}\|D^2\phi(x)\|\varepsilon^{1-\alpha} \right) + \frac{1}{2}\varepsilon^{2-2\alpha}(\|D^2\phi(x)\| + \varepsilon^{-\gamma}) - 4\varepsilon^\sigma M_\varepsilon^x[\phi] \\ &\quad + \frac{1}{2}\varepsilon^\sigma\lambda - \varepsilon^2\mathcal{F}(t, x, z, p, \Gamma_{\text{opt}} - \lambda I). \end{aligned} \quad (1.4.49)$$

Since  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  in case (iv), we deduce from the assumption (1.4.43) that

$$\varepsilon^{1-\alpha} - \text{dist}(x) \geq \varepsilon^\rho \gg \varepsilon^\sigma. \quad (1.4.50)$$

Since  $M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}$  and  $\varepsilon^{2-2\alpha-\gamma} \ll \varepsilon^{1-\alpha-\kappa+\rho}$  using (1.4.24), we conclude by (1.4.50) that

$$\begin{aligned} \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) \left( M_\varepsilon^x[\phi] + \frac{9}{2}\|D^2\phi(x)\|\varepsilon^{1-\alpha} \right) &+ \frac{1}{2}\varepsilon^{2-2\alpha}(\|D^2\phi(x)\| + \varepsilon^{-\gamma}) - 4\varepsilon^\sigma M_\varepsilon^x[\phi] \\ &\leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi]. \end{aligned} \quad (1.4.51)$$

It remains to control the terms in (1.4.49) depending on  $\lambda$ . If  $-1 \leq \lambda \leq -\varepsilon^\alpha$ , then  $\varepsilon^{2\sigma}\lambda \leq -\varepsilon^{2\sigma+\alpha}$  and  $\mathcal{F}(t, x, z, p, \Gamma_{\text{opt}} - \lambda I)$  is bounded. So for such  $\lambda$  we have

$$\frac{1}{2}\varepsilon^{2\sigma}\lambda - \varepsilon^2\mathcal{F}(t, x, z, p, \Gamma_{\text{opt}} - \lambda I) \leq -\frac{1}{2}\varepsilon^{2\sigma+\alpha} + O(\varepsilon^2). \quad (1.4.52)$$

In this case, the right-hand side is  $\leq -\varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}})$  when  $\varepsilon$  is sufficiently small since  $\varepsilon^{2\sigma+\alpha} \gg \varepsilon^2$  by (1.4.43).

To complete case (b), suppose  $\lambda \leq -1$ . Then using the growth hypothesis (1.1.7) and recalling that  $p$  is near  $p_{\text{opt}}$  we have

$$\frac{1}{2}\varepsilon^{2\sigma}\lambda - \varepsilon^2 \mathcal{F}(t, x, z, p, D^2\phi(x) - \lambda I) \leq -\frac{1}{2}\varepsilon^{2\sigma}|\lambda| + C\varepsilon^2(1 + |\lambda|^r). \quad (1.4.53)$$

Now notice that  $|\lambda| \leq C(1 + \|\Gamma\|) \leq C\varepsilon^{-\gamma}$ . Since  $\gamma(r-1) < 2 - 2\sigma$  by (1.4.43), we have  $\varepsilon^{2\sigma}|\lambda| \gg \varepsilon^2|\lambda|^r$ . Therefore

$$-\frac{1}{2}\varepsilon^{2\sigma}|\lambda| + C\varepsilon^2|\lambda|^r \leq -\frac{1}{4}\varepsilon^{2\sigma} \leq -\varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}),$$

for  $\varepsilon$  small enough. Case (b) is now complete.

Finally in case (c), we take  $\Delta\hat{x}$  to be parallel to  $p_{\text{opt}}^M - p$  with norm  $\varepsilon^{1-\alpha}$ , and with the sign chosen such that

$$(p_{\text{opt}}^M - p) \cdot \Delta\hat{x} = -\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| \leq -3(\varepsilon^{1-\alpha} - \text{dist}(x))|M_\varepsilon^x[\phi]| \leq -3\varepsilon^{1-\alpha-\kappa+\rho}. \quad (1.4.54)$$

Estimating the other terms on the left-hand side of (1.4.36), some manipulations analogous to those made in Lemma 1.4.11 led us to

$$|(D\phi(x) - p_{\text{opt}}^M) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi]| \leq \frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) (3|M_\varepsilon^x[\phi]| + 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}).$$

From (1.4.54), we deduce that

$$(D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] \leq -\frac{1}{2}\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| + 2\|D^2\phi(x)\|\varepsilon^{2-2\alpha}.$$

Estimating the other terms

$$|\langle (\Gamma_{\text{opt}}(x) - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle| \leq (C + \|\Gamma\|)\|\Delta\hat{x}\|^2 \leq C\varepsilon^{-\gamma+2-2\alpha}, \quad (1.4.55)$$

and

$$\varepsilon^2|\mathcal{F}(t, x, z, p, \Gamma)| \leq C\varepsilon^2(1 + \|p\|^q + \|\Gamma\|^r) \leq C(\varepsilon^2 + \varepsilon^2\|p\|^q + \varepsilon^{2-\gamma r}). \quad (1.4.56)$$

Thus

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 f(t, x, z, p, \Gamma) \\ \leq -\frac{1}{2}\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| + C\varepsilon^2\|p\|^q + O(\varepsilon^{2-2\alpha} + \varepsilon^{-\gamma+2-2\alpha} + \varepsilon^{2-\gamma r}). \end{aligned}$$

Since  $\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| \geq 2\varepsilon^{1-\alpha-\kappa+\rho}$  by using (1.4.54), we obtain that

$$\varepsilon^{-\gamma+2-2\alpha} + \varepsilon^{2-\gamma r} \ll \varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\|, \quad (1.4.57)$$

noticing that  $\min(-\gamma + 2 - 2\alpha, 2 - \gamma r) > 1 - \alpha - \kappa + \rho$  by using (1.2.15) and (1.4.24). Thus, by combining (1.4.55)–(1.4.57), we conclude that

$$\begin{aligned} (D\phi(x) - p) \cdot \Delta\hat{x} + \|\Delta\hat{x} - \Delta x\| M_\varepsilon^x[\phi] + \frac{1}{2}\langle D^2\phi(x)\Delta x, \Delta x \rangle - \frac{1}{2}\langle \Gamma\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \\ \leq -\frac{1}{2\sqrt{2}}\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| + C\varepsilon^2\|p\|^q. \end{aligned}$$

If  $\|p\| \leq 2\|p_{\text{opt}}^M\|$ , then  $\varepsilon^2\|p\|^q \ll \varepsilon^{1-\alpha-\kappa+\rho}$ . If  $\|p\| \geq 2\|p_{\text{opt}}^M\|$ , we infer from the condition on  $\beta$  in (1.2.15) that  $\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| \sim \varepsilon^{1-\alpha}\|p\| \gg \varepsilon^2\|p\|^q$ . In either case the term  $\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\|$  dominates and we get

$$(p_{\text{opt}}^M - p) \cdot \Delta\hat{x} + \frac{1}{2}\langle (\Gamma_{\text{opt}}^M - \Gamma)\Delta\hat{x}, \Delta\hat{x} \rangle - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \leq -\frac{1}{4}\varepsilon^{1-\alpha}\|p_{\text{opt}}^M - p\| \leq \frac{3}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi].$$

The right-hand side of this inequality is certainly  $\leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2 \min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}})$  when  $\varepsilon$  is small. Case (c) is now complete which finishes the proof of Proposition 1.4.10.

### 1.4.3 Application to stability

To prove stability in Section 1.5, we will need some global variants of Propositions 1.4.5 and 1.4.10. It is at this point that the uniformity of the constants in (1.1.6)–(1.1.7) in  $x$  and  $t$ , and the growth condition (1.1.7) intervene. We must also take care of the Neumann boundary condition. Unlike the Dirichlet problem solved in [88], it is no longer appropriate to consider constant functions as test functions. For this reason, we are going to consider a  $C_b^2(\overline{\Omega})$ -function  $\psi$  such that

$$\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1 \quad \text{on } \partial\Omega. \quad (1.4.58)$$

It is worth noticing that  $\psi$  has exactly the same properties as the function introduced in Section 1.2.3 for the game associated to the elliptic PDE with Neumann boundary condition. If we take  $\psi = (\|h\|_{L^\infty} + 1)\psi_1$  where  $\psi_1 \in C_b^2(\overline{\Omega})$  such that  $\frac{\partial \psi_1}{\partial n} = 1$  on  $\partial\Omega$ , it is clear that  $\|\psi\|_{C_b^2(\overline{\Omega})} = \|\psi_1\|_{C_b^2(\overline{\Omega})}(1 + \|h\|_{L^\infty})$ .

The next lemma is the crucial point to obtain stability in both parabolic and elliptic settings.

**Lemma 1.4.14.** *If  $\psi \in C_b^2(\overline{\Omega})$  satisfies (1.4.58), then there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon < \varepsilon_0$  and for all  $x \in \Omega(\varepsilon^{1-\alpha})$ ,*

$$-\|h\|_{L^\infty} - \|D\psi\|_{L^\infty(\overline{\Omega})} \leq M_\varepsilon^x[\psi] \leq -\frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq m_\varepsilon^x[-\psi] \leq \|h\|_{L^\infty} + \|D\psi\|_{L^\infty(\overline{\Omega})}. \quad (1.4.59)$$

*Proof.* We shall demonstrate the bounds on  $M_\varepsilon^x[\psi]$  in (1.4.59); the proof for  $m_\varepsilon^x[-\psi]$  is entirely parallel. The left-hand side inequality is clear by the Cauchy-Schwarz inequality. Let us consider  $0 < \varepsilon < \varepsilon_0$ , where  $\varepsilon_0 = \left(4\|D^2\psi\|_{L^\infty(\overline{\Omega})} + 2\right)^{-\frac{1}{1-\alpha}}$ . By the geometric relation (1.4.4), we observe that every move  $\Delta x$  associated to the move  $\Delta \hat{x}$  decided by Mark satisfies

$$\|\Delta x\| \leq 2\varepsilon^{1-\alpha} \leq (2\|D^2\psi\|_{L^\infty(\overline{\Omega})} + 1)^{-1}.$$

By the Cauchy-Schwarz inequality and using that  $\psi \in C_b^2(\overline{\Omega})$ , we have

$$\begin{aligned} h(x + \Delta x) - D\psi(x) \cdot n(x + \Delta x) &\leq \|h\|_{L^\infty} - D\psi(x + \Delta x) \cdot n(x + \Delta x) + (D\psi(x + \Delta x) - D\psi(x)) \cdot n(x + \Delta x) \\ &\leq -1 + \|D^2\psi\|_{L^\infty(\overline{\Omega})}\|\Delta x\| \leq -\frac{1}{2}. \end{aligned}$$

Then, by passing to the sup, we get the desired result.  $\square$

**Lemma 1.4.15.** *Let  $\phi \in C_b^2(\overline{\Omega})$ . Assume that  $p_{opt}^m$ ,  $p_{opt}^M$  and  $\Gamma_{opt}$  are the strategies, associated to  $\phi$ , respectively defined by (1.4.13), (1.4.14) and (1.4.15). Then, for all  $x \in \Omega(\varepsilon^{1-\alpha})$ , we have*

$$\max(\|p_{opt}^m(x)\|, \|p_{opt}^M(x)\|) \leq \frac{1}{2} \left( \|h\|_{L^\infty} + 3\|D\phi\|_{C_b^1(\overline{\Omega})} \right) \quad \text{and} \quad \|\Gamma_{opt}(x)\| \leq \frac{3}{2}\|D^2\phi\|_{L^\infty(\overline{\Omega})}.$$

*Proof.* The proof being exactly the same for  $p_{opt}^m$ , it is sufficient to show the result for  $p_{opt}^M$ . By the triangle inequality and (1.4.14), we have

$$\begin{aligned} \|p_{opt}^M(x) - D\phi(x)\| &\leq \frac{1}{2} \left( 1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) \left( |M_\varepsilon^x[\phi]| + \frac{1}{2}\varepsilon^{1-\alpha} \left( 1 + \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}} \right) \|D^2\phi(x)\| \right) \\ &\leq \frac{1}{2} (|M_\varepsilon^x[\phi]| + \varepsilon^{1-\alpha}\|D^2\phi(x)\|). \end{aligned}$$

Since  $M_\varepsilon^x[\phi]$  is  $\varepsilon, x$ -bounded by  $\|h\|_{L^\infty} + \|D\phi\|_{L^\infty(\overline{\Omega})}$ , we deduce the desired inequality on  $\|p_{opt}^M(x)\|$ . Similarly, the estimate on  $\|\Gamma_{opt}(x)\|$  stems directly from (1.4.15) and the triangle inequality.  $\square$

In preparation for stability, we need to compute the action of  $S_\varepsilon$  on  $\psi$ . According to Lemma 1.4.14, only some cases proposed in Proposition 1.4.10 must be considered. The next proposition gives the required estimates for  $S_\varepsilon$  concerning these cases.

**Proposition 1.4.16.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma, \rho, \kappa$  fulfill (1.2.13)–(1.2.15) and (1.4.23)–(1.4.24). Then for any  $x, t, z$  and any  $C_b^2(\overline{\Omega})$ -function  $\phi$  defined near  $x$ ,  $S_\varepsilon[x, t, z, \phi]$  being defined by (1.4.1), we have*

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \leq \begin{cases} C\varepsilon^2(1 + |z|), & \text{if } \text{dist}(x) \geq \varepsilon^{1-\alpha}, \\ 3\varepsilon^{1-\alpha}M_\varepsilon^x[\phi] + C\varepsilon^2(1 + |z|), & \text{if } \text{dist}(x) \leq \varepsilon^{1-\alpha} \text{ and } M_\varepsilon^x[\phi] \geq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}, \\ C\varepsilon^2(1 + |z|), & \text{if } \varepsilon^{1-\alpha} - \varepsilon^\rho \leq \text{dist}(x) \leq \varepsilon^{1-\alpha} \text{ and } M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}, \\ \frac{1}{4}\varepsilon^\rho M_\varepsilon^x[\phi] + C\varepsilon^2(1 + |z|), & \text{if } \text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho \text{ and } M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}, \end{cases} \quad (1.4.60)$$

with a constant  $C$  that depends on  $\|D\phi\|_{C_b^1(\overline{\Omega})} + \|h\|_{L^\infty}$  but is independent of  $x, t$  and  $z$ .

Moreover, if  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ , or if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $m_\varepsilon^x[\phi] > \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , then

$$-C\varepsilon^2(1 + |z|) \leq S_\varepsilon[x, t, z, \phi] - \phi(x), \quad (1.4.61)$$

with a constant  $C$  that depends on  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  but is independent of  $x, t$  and  $z$ .

*Proof.* The arguments in the different cases are the same as those given in the proof of Proposition 1.4.10 but we must pay attention to the uniformity of the constant. For the second part, since  $\mathcal{F}$  grows linearly by (1.1.5) and  $\|(D\phi(x), D^2\phi(x))\| \leq \|D\phi\|_{C_b^1(\overline{\Omega})}$ , we have

$$|\mathcal{F}(t, x, z, D\phi(x), D^2\phi(x))| \leq C(1 + |z|), \quad (1.4.62)$$

with a constant  $C$  that depends on  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  but is independent of  $x, t$  and  $z$ . The lower bound

$$S_\varepsilon[x, t, z, \phi] - \phi(x) \geq -\varepsilon^2\mathcal{F}(x, t, z, D\phi(x), D^2\phi(x)) \geq -C\varepsilon^2(1 + |z|)$$

is a consequence of Proposition 1.4.5 and (1.4.62).

Similarly, since we know by Lemma 1.4.15 that  $\max(\|p_{\text{opt}}^m(x)\|, \|p_{\text{opt}}^M(x)\|) + \|\Gamma_{\text{opt}}(x)\|$  is uniformly bounded by  $\frac{1}{2}\|h\|_{L^\infty} + 3\|D\phi\|_{C_b^1(\overline{\Omega})}$ , we get that

$$\max(|\mathcal{F}(t, x, z, p_{\text{opt}}^m(x), \Gamma_{\text{opt}}(x))|, |\mathcal{F}(t, x, z, p_{\text{opt}}^M(x), \Gamma_{\text{opt}}(x))|) \leq C(1 + |z|), \quad (1.4.63)$$

with a constant  $C$  that depends on  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  and  $\|h\|_{L^\infty}$  but is independent of  $x, t$  and  $z$ .

We shall prove the estimate for the fourth alternative of (1.4.60) by examining the proof of Proposition 1.4.10 case (iv), the proofs for the other alternatives being quite similar. Since  $\mathcal{F}$  is locally Lipschitz by (1.1.6),

$$\min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) \geq \mathcal{F}(t, x, z, p_{\text{opt}}^M(x), \Gamma_{\text{opt}}(x)) - C(1 + |z|) \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) (\|h\|_{L^\infty} + \|D\phi\|_{L^\infty(\overline{\Omega})}),$$

where  $C$  depends only on  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  and  $\|h\|_{L^\infty}$  by the estimates on  $p_{\text{opt}}^M$  and  $\Gamma_{\text{opt}}$  given by Lemma 1.4.15. By using (1.4.63), we deduce that there exists a constant  $C$  depending only on  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  and  $\|h\|_{L^\infty}$  such that

$$\min_{p \in B(p_{\text{opt}}^M, r)} \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}) \geq -C(1 + |z|). \quad (1.4.64)$$

In case (a), by combining (1.4.64) and the locally Lipschitz character (1.1.6) of  $\mathcal{F}$  on  $\Gamma$ , the estimate (1.4.39) gets replaced by

$$\mathcal{F}(t, x, z, p, \Gamma) \geq -C(1 + |z|)(1 + \varepsilon^\alpha),$$

whence by (1.4.63) there exists a constant  $C$  depending on  $\|D\phi\|_{C_b^1(\overline{\Omega})} + \|h\|_{L^\infty}$  such that

$$-\varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma) \leq C(1 + |z|)\varepsilon^2.$$

In case (b), since the domain satisfies both the uniform interior and exterior ball conditions, we notice that the constant  $C_1$  corresponding to the curvature of the boundary (see Lemma 1.4.2) is  $x$ -bounded. This implies that the first order estimate (1.4.45) is valid independently of  $x$  for  $\varepsilon$  sufficiently small. Thus, the estimate (1.4.51) is valid uniformly in  $x$ . Besides, the estimate (1.4.52) gets replaced by

$$\frac{1}{2}\varepsilon^{2\sigma}\lambda - \varepsilon^2 \mathcal{F}(t, x, z, p, \Gamma_{\text{opt}}(x) - \lambda I) \leq -\frac{1}{2}\varepsilon^{2\sigma+\alpha} + C\varepsilon^2(1 + |z|)\|p\|\|\Gamma_{\text{opt}}(x) - \lambda I\|,$$

where  $C$  depends on  $\|D\phi\|_{C_b^1(\overline{\Omega})} + \|h\|_{L^\infty}$ . We obtain an estimate of the desired form by dropping the first term and observing that  $\lambda$  is bounded. In second half of case (b) and in case (c) we used the growth estimate (1.1.7); since  $z$  enters linearly on the right-hand side of (1.1.7), the previous calculation still applies but we get an additional term of the form  $C|z|\varepsilon^2$  in (1.4.53)–(1.4.56).  $\square$

The following corollary provides the key estimate for stability in the parabolic setting.

**Corollary 1.4.17.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.6)–(1.1.7) and assume  $\alpha, \beta, \gamma$  fulfill (1.2.13)–(1.2.15). Then, for any  $x, t, z$  and  $\psi \in C_b^2(\overline{\Omega})$  satisfying (1.4.58), we have*

$$S_\varepsilon[x, t, z, \psi] - \psi(x) \leq C(1 + |z|)\varepsilon^2 \quad \text{and} \quad S_\varepsilon[x, t, z, -\psi] - (-\psi)(x) \geq -C(1 + |z|)\varepsilon^2, \quad (1.4.65)$$

with a constant  $C$  that is independent of  $x, t, z$  but depends on  $\|D\psi\|_{C_b^1(\overline{\Omega})}$  and  $\|h\|_{L^\infty}$ .

*Proof.* We shall prove the first estimate, the second follows exactly the same lines. By Lemma 1.4.14, we have  $M_\varepsilon^x[\psi] \leq -\frac{1}{2}$  for all  $x \in \Omega(\varepsilon^{1-\alpha})$ . We introduce  $\rho$  fulfilling (1.4.23). By putting together the estimates obtained from (1.4.61) and the third alternative in (1.4.60), we get that there exists a constant  $C$  depending only on  $\|D\psi\|_{C_b^1(\overline{\Omega})}$  and  $\|h\|_{L^\infty}$  such that

$$S_\varepsilon[x, t, z, \psi] - \psi(x) \leq \begin{cases} C\varepsilon^2(1 + |z|), & \text{if } \text{dist}(x) \geq \varepsilon^{1-\alpha} - \varepsilon^\rho, \\ \frac{1}{4}\varepsilon^\rho M_\varepsilon^x[\psi] + C\varepsilon^2(1 + |z|), & \text{if } \text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho. \end{cases}$$

Noticing that  $M_\varepsilon^x[\psi]$  is negative, we get the proposed result.  $\square$

#### 1.4.4 The elliptic case

For the game corresponding to the stationary equation, we consider the operator  $Q_\varepsilon$  defined for any  $x \in \overline{\Omega}$ ,  $z \in \mathbb{R}$ , and any continuous function  $\phi: \overline{\Omega} \rightarrow \mathbb{R}$ , by

$$Q_\varepsilon[x, z, \phi] = \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \left[ e^{-\lambda \varepsilon^2} \phi(x + \Delta x) - \left( p \cdot \Delta \hat{x} + \frac{1}{2} \langle \Gamma \Delta \hat{x}, \Delta \hat{x} \rangle + \varepsilon^2 \mathcal{F}(x, z, p, \Gamma) - \|\Delta \hat{x} - \Delta x\| h(x + \Delta x) \right) \right], \quad (1.4.66)$$

with the usual conventions that  $p, \Gamma$  and  $\Delta \hat{x}$  are constrained by (1.2.16) and (1.2.17) and that  $\Delta x$  is determined by (1.2.21). We can easily check that the operator  $Q_\varepsilon$  is still monotone but its action on shifted functions by a constant is described by the following way: for all function  $\phi \in C(\overline{\Omega})$  and  $c \in \mathbb{R}$ ,

$$Q_\varepsilon[x, z, c + \phi] = e^{-\lambda \varepsilon^2} c + Q_\varepsilon[x, z, \phi]. \quad (1.4.67)$$

The dynamic programming inequalities (1.2.61)–(1.2.62) can be concisely written as

$$u^\varepsilon(x) \leq Q_\varepsilon[x, u^\varepsilon(x), u^\varepsilon] \quad \text{and} \quad v^\varepsilon(x) \geq Q_\varepsilon[x, v^\varepsilon(x), v^\varepsilon].$$

In the elliptic setting, we can formally derive the PDE by following the same lines as for the parabolic framework. We keep the optimal strategies  $p_{\text{opt}}^m$ ,  $p_{\text{opt}}^M$  and  $\Gamma_{\text{opt}}$  for Helen, defined by (1.4.13), (1.4.14) and (1.4.15) in an orthonormal basis  $\mathcal{B} = (e_1 = n(\bar{x}), e_2, \dots, e_d)$ . The next proposition is the elliptic analogue of Propositions 1.4.5 and 1.4.10. It establishes the consistency estimates for  $Q_\varepsilon$  defined by (3.1.8).

**Proposition 1.4.18.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.9)–(1.1.10) and assume  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\rho$  and  $\kappa$  fulfill (1.2.13)–(1.2.15) and (1.4.23)–(1.4.24). Let  $p_{\text{opt}}^m$ ,  $p_{\text{opt}}^M$  and  $\Gamma_{\text{opt}}$  be respectively defined in the orthonormal basis  $\mathcal{B} = (e_1 = n(\bar{x}), e_2, \dots, e_d)$  by (1.4.13)–(1.4.15). For any  $x$ ,  $z$  and any smooth function  $\phi$  defined near  $x$ , we distinguish two cases for the lower bound estimate:*

- i. *Big bonus: if  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$  or  $m_\varepsilon^x[\phi] > \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , then*

$$-\varepsilon^2(\mathcal{F}(x, z, D\phi(x), D^2\phi(x)) + \lambda\phi(x)) \leq Q_\varepsilon[x, z, \phi] - \phi(x). \quad (1.4.68)$$

- ii. *Penalty or small bonus: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $m_\varepsilon^x[\phi] \leq \frac{1}{2}(3\varepsilon^{1-\alpha} - \text{dist}(x))\|D^2\phi(x)\|$ , then*

$$\frac{1}{2}(\varepsilon^{1-\alpha} - \text{dist}(x)) (sm_\varepsilon^x[\phi] - 4\|D^2\phi(x)\|\varepsilon^{1-\alpha}) - \varepsilon^2(\mathcal{F}(x, z, p_{\text{opt}}^m(x), \Gamma_{\text{opt}}(x)) + \lambda\phi(x)) \leq Q_\varepsilon[x, z, \phi] - \phi(x),$$

where  $s = -1$  if  $m_\varepsilon^x[\phi] \geq 0$  and  $s = 3$  if  $m_\varepsilon^x[\phi] < 0$ .

For the upper bound estimate, we distinguish four cases:

- i. *Big bonus: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $M_\varepsilon^x[\phi] > \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , then*

$$Q_\varepsilon[x, z, \phi] - \phi(x) \leq 3(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2(\mathcal{F}(x, z, p_{\text{opt}}^M(x), \Gamma_{\text{opt}}(x)) + \lambda\phi(x)) + o(\varepsilon^2).$$

- ii. *Far from the boundary with a small bonus: if  $\varepsilon^{1-\alpha} - \varepsilon^\rho \leq \text{dist}(x) \leq \varepsilon^{1-\alpha}$  and  $M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , or if  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ , then*

$$Q_\varepsilon[x, z, \phi] - \phi(x) \leq -\varepsilon^2(\mathcal{F}(x, z, D\phi(x), D^2\phi(x)) + \lambda\phi(x)) + o(\varepsilon^2). \quad (1.4.69)$$

- iii. *Close to the boundary with a small bonus/penalty: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  and  $-\varepsilon^{1-\alpha-\kappa} \leq M_\varepsilon^x[\phi] \leq \frac{4}{3}\|D^2\phi(x)\|\varepsilon^{1-\alpha}$ , then*

$$Q_\varepsilon[x, z, \phi] - \phi(x) \leq -\varepsilon^2(\mathcal{F}(x, z, D\phi(x), D^2\phi(x)) + C_1 I + \lambda\phi(x)) + o(\varepsilon^2),$$

with  $C_1 = \frac{20}{3}\|D^2\phi(x)\| \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right)$ .

- iv. *Close to the boundary with a big bonus: if  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  and  $M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}$ , then*

$$Q_\varepsilon[x, z, \phi] - \phi(x) \leq \frac{1}{4}(\varepsilon^{1-\alpha} - \text{dist}(x))M_\varepsilon^x[\phi] - \varepsilon^2 \left( \min_{p \in B(p_{\text{opt}}^M(x), r)} \mathcal{F}(x, z, p, \Gamma_{\text{opt}}(x)) + \lambda\phi(x) \right) + o(\varepsilon^2), \quad (1.4.70)$$

with  $r$  defined by  $r := 3 \left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\phi]|$ .

Moreover the implicit constants in the error term are uniform as  $x$  and  $z$  range over a compact subset of  $\bar{\Omega} \times \mathbb{R}$ .

*Proof.* The arguments are entirely parallel to the proofs of Propositions 1.4.5 and 1.4.10. □

For stability we will need a variant of the preceding lemma. This is where we use the hypothesis (1.1.8) on the  $z$ -dependence of  $\mathcal{F}$ .

**Lemma 1.4.19.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.8)–(1.1.10) and assume as always that  $\alpha, \beta, \gamma, \rho$  and  $\kappa$  satisfy (1.2.13)–(1.2.15) and (1.4.23)–(1.4.24). Let  $\psi \in C_b^2(\overline{\Omega})$  satisfy (1.2.55). Fix  $M$  and  $m$  two positive constants such that  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})} \leq M$ . Then, there exists  $C_* = C_*(\|D\psi\|_{C_b^1(\overline{\Omega})}, \|h\|_{L^\infty})$  such that for any  $|z| \leq M$  and any  $x \in \overline{\Omega}$ , we have*

$$Q_\varepsilon[x, z, m + \psi] - (m + \psi(x)) \leq \varepsilon^2 (1 + (\lambda - \eta)|z| + C_*) - \lambda\varepsilon^2 (m + \psi(x)),$$

and

$$Q_\varepsilon[x, z, -m - \psi] - (-m - \psi(x)) \geq -\varepsilon^2 (1 + (\lambda - \eta)|z| + C_*) - \lambda\varepsilon^2 (-m - \psi(x)),$$

for all sufficiently small  $\varepsilon$  (the smallness condition on  $\varepsilon$  depends on  $M$ , but not on  $x$ ).

Moreover, if  $\phi \in C_b^2(\overline{\Omega})$ , then there exists  $C = C(M, \|D\phi\|_{C_b^1(\overline{\Omega})}, \|h\|_{L^\infty})$  such that for any  $|z| \leq M$  and any  $x \in \overline{\Omega}$  such that  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$  and  $M_\varepsilon^x[\phi] \leq -\varepsilon^{1-\alpha-\kappa}$ ,

$$Q_\varepsilon[x, z, \phi] - \phi(x) \leq \frac{1}{4} (\varepsilon^{1-\alpha} - \text{dist}(x)) M_\varepsilon^x[\phi] + C\varepsilon^2 - \lambda\varepsilon^2 \phi(x), \quad (1.4.71)$$

for all sufficiently small  $\varepsilon$  (the smallness condition on  $\varepsilon$  depends on  $M$ , but not on  $x$ ).

*Proof.* We shall prove the first inequality, the proof of the second being entirely parallel. The assumption  $|z| \leq M$  ensures that the constants in (1.1.9) and (1.1.10) are uniform. Then the implicit constants in the error terms of (1.4.69) and (1.4.70) are  $x, z$ -uniform for  $\varepsilon$  small enough, and the smallness condition depends only on  $M$ . Since  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})} \leq M$  we can use the dynamic programming inequalities (1.2.61)–(1.2.62). First of all, by the action of  $Q_\varepsilon$  on constant functions provided by (1.4.67), we have

$$Q_\varepsilon[x, z, m + \psi] - (m + \psi(x)) = (e^{-\lambda\varepsilon^2} - 1)m + Q_\varepsilon[x, z, \psi] - \psi(x),$$

and noticing that  $e^{-\lambda\varepsilon^2} m = (1 - \lambda\varepsilon^2)m + O(\varepsilon^4 m)$ , it is sufficient to get the estimate corresponding to  $m = 0$ . By Lemma 1.4.14, we observe that every  $x \in \Omega(\varepsilon^{1-\alpha})$  satisfies  $M_\varepsilon^x[\psi] \leq -\frac{1}{2}$ . We now need to distinguish two cases according to the distance to the boundary by introducing  $\rho$  fulfilling (1.4.23). If  $x \in \overline{\Omega}$  such that  $\text{dist}(x) \geq \varepsilon^{1-\alpha} - \varepsilon^\rho$ , since  $\|(D\psi(x), D^2\psi(x))\| \leq K_1 = \|D\psi\|_{C_b^1(\overline{\Omega})}$ , we deduce by Assumption (1.1.8) on  $\mathcal{F}$  that there exists  $C_{K_1}^*$  such that for all  $x$  we have

$$|\mathcal{F}(x, z, D\psi(x), D^2\psi(x))| \leq (\lambda - \eta)|z| + C_{K_1}^*,$$

which gives by (1.4.69) that for all  $x \in \overline{\Omega}$  such that  $\text{dist}(x) \geq \varepsilon^{1-\alpha}$ ,

$$Q_\varepsilon[x, z, \psi] - \psi(x) \leq \varepsilon^2 ((\lambda - \eta)|z| + C_{K_1}^*) - \lambda\varepsilon^2 \psi(x) + o(\varepsilon^2). \quad (1.4.72)$$

If  $x \in \overline{\Omega}$  such that  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$ , combining the triangle inequality with the inequalities given by Lemma 1.4.15 gives that, for all  $p \in B(p_{\text{opt}}^M(x), r)$  with  $r = 3\left(1 - \frac{\text{dist}(x)}{\varepsilon^{1-\alpha}}\right) |M_\varepsilon^x[\psi]|$ ,

$$\|(p, \Gamma_{\text{opt}}(x))\| \leq \|p_{\text{opt}}^M(x)\|_{L^\infty} + r + \|\Gamma_{\text{opt}}(x)\|_{L^\infty} \leq K_2 = \frac{7}{2} \|h\|_{L^\infty} + 6 \|D\psi\|_{C_b^1(\overline{\Omega})},$$

since  $M_\varepsilon^x[\psi]$  is  $\varepsilon, x$ -bounded by  $\|h\|_{L^\infty} + \|D\psi\|_{L^\infty}$ . The assumption (1.1.8) on  $\mathcal{F}$  yields that there exists  $C_{K_2}^*$  such that,

$$\left| \min_{p \in B(p_{\text{opt}}^M(x), r)} \mathcal{F}(x, z, p, \Gamma_{\text{opt}}(x)) \right| \leq (\lambda - \eta)|z| + C_{K_2}^*, \quad (1.4.73)$$

By using this inequality in (1.4.70) and recalling that  $M_\varepsilon^x[\psi] \leq -\frac{1}{2}$ , we conclude that, for all  $x \in \overline{\Omega}$  such that  $\text{dist}(x) \leq \varepsilon^{1-\alpha} - \varepsilon^\rho$ ,

$$Q_\varepsilon[x, z, \psi] - \psi(x) \leq \varepsilon^2 ((\lambda - \eta)|z| + C_{K_2}^*) - \lambda\varepsilon^2 \psi(x) + o(\varepsilon^2). \quad (1.4.74)$$

By comparing (1.4.72) and (1.4.74) we get the desired result by taking  $C_* = \max(C_{K_1}^*, C_{K_2}^*)$ .

To prove the third inequality, it is sufficient to replace the assumption (1.1.8) by (1.1.10) in the previous estimates. For instance, instead of (1.4.73), there exists a constant  $C$  depending only on  $M$ ,  $\|h\|_{L^\infty}$ , and  $\|D\phi\|_{C_b^1(\overline{\Omega})}$  such that  $\left| \min_{p \in B(p_{\text{opt}}^M(x), r)} \mathcal{F}(x, z, p, \Gamma_{\text{opt}}(x)) \right| \leq C$ . The rest of the proof remains unchanged.  $\square$

## 1.5 Stability

In the time-dependent setting, we showed in Section 1.3.2 that if  $v^\varepsilon$  and  $u^\varepsilon$  remain bounded as  $\varepsilon \rightarrow 0$  then  $\underline{v}$  is a supersolution and  $\bar{u}$  is a subsolution. The argument was local, using mainly the consistency of the game as a numerical scheme. It remains to prove that  $v^\varepsilon$  and  $u^\varepsilon$  are indeed bounded; this is achieved in Section 1.5.1.

For the stationary setting, we must do more. Even the existence of  $U^\varepsilon(x, z)$  remains to be proved. We also need to show that the associated functions  $u^\varepsilon$  and  $v^\varepsilon$  are bounded, away from  $M$ , so that we can apply the dynamic programming inequalities at each  $x \in \overline{\Omega}$ . These goals will be achieved in Section 1.5.2, provided the parameters  $M$  and  $m$  satisfy (i)  $m = M - 1 - 2\|\psi\|_{L^\infty}$  and (ii)  $M$  is sufficiently large. We also show in Section 1.5.2 that if  $\mathcal{F}$  is a nondecreasing function on  $z$  then  $U^\varepsilon$  is strictly decreasing on  $z$ . As a consequence, this result implies that  $\underline{v} \leq \bar{u}$ , allowing us to conclude that  $\underline{v} = \bar{u}$  is the unique viscosity solution if the boundary value problem has a comparison principle.

### 1.5.1 The parabolic case

To obtain stability, we are going to consider one more time a  $C_b^2(\overline{\Omega})$ -function  $\psi$  such that

$$\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1$$

in order to take care of the Neumann boundary condition.

**Proposition 1.5.1.** *Assume the hypotheses of Propositions 1.4.5 and 1.4.10 hold, and suppose furthermore that the final-time data are uniformly bounded:*

$$|g(x)| \leq B \quad \text{for all } x \in \overline{\Omega}.$$

*Then there exists a constant  $s = s(\|\psi\|_{C_b^2(\overline{\Omega})})$ , independent of  $\varepsilon$ , such that*

$$u^\varepsilon(x, t) \leq (B + \|\psi\|_{L^\infty(\overline{\Omega})})s^{T-t} + \psi(x) \quad \text{for all } x \in \overline{\Omega},$$

*and*

$$v^\varepsilon(x, t) \geq -(B + \|\psi\|_{L^\infty(\overline{\Omega})})s^{T-t} - \psi(x) \quad \text{for all } x \in \overline{\Omega},$$

*for every  $t < T$ .*

*Proof.* We shall demonstrate the lower bound on  $v^\varepsilon$ ; the proof of the upper bound on  $u^\varepsilon$  is entirely parallel. The argument proceeds backward in time  $t_k = T - k\varepsilon^2$ . At  $k = 0$ , we have a uniform bound  $v^\varepsilon(x, T) = g(x) \geq -B$  by hypothesis, and we may assume without loss of generality that  $B \geq 1$ . Since  $\psi$  is bounded on  $\overline{\Omega}$ , we can suppose that

$$v^\varepsilon(x, T) = g(x) \geq -B_0 - \psi(x),$$

where  $B_0 = B + \|\psi\|_{L^\infty(\overline{\Omega})}$ . Now suppose that for fixed  $k \geq 0$  we already know a bound  $v^\varepsilon(\cdot, t_k) \geq -B_k - \psi$ . By the dynamic programming inequality (1.2.24), we have

$$v^\varepsilon(x, t_k - \varepsilon^2) \geq S_\varepsilon[x, t, v^\varepsilon(x, t_k - \varepsilon^2), v^\varepsilon(\cdot, t_k)].$$



Since  $S_\varepsilon$  is monotone in its last argument, we have

$$v^\varepsilon(x, t_k - \varepsilon^2) \geq S_\varepsilon[x, t, v^\varepsilon(x, t_k - \varepsilon^2), -B_k - \psi].$$

By applying successively (1.4.2) and Corollary 1.4.17, we deduce that

$$\begin{aligned} S_\varepsilon[x, t, v^\varepsilon(x, t_k - \varepsilon^2), -B_k - \psi] &= -B_k + S_\varepsilon[x, t, v^\varepsilon(x, t_k - \varepsilon^2), -\psi] \\ &\geq -B_k - \psi(x) - C(1 + |v^\varepsilon(x, t_k - \varepsilon^2)|)\varepsilon^2, \end{aligned}$$

where  $C$  depends only on  $\|D\psi\|_{C_b^1(\overline{\Omega})}$ . If  $v^\varepsilon(x, t_k - \varepsilon^2) \geq 0$ , then it is over (recall we are looking for a lower bound  $-B_{k+1} \leq -1$ ). Otherwise, we have

$$(1 - C\varepsilon^2)v^\varepsilon(x, t_k - \varepsilon^2) \geq -B_k - C\varepsilon^2 - \psi(x).$$

By dividing by  $1 - C\varepsilon^2$ , we get

$$v^\varepsilon(x, t_k - \varepsilon^2) \geq -\frac{B_k + C\varepsilon^2}{1 - C\varepsilon^2} - \frac{1}{1 - C\varepsilon^2}\psi(x) = -\frac{B_k + C\varepsilon^2(1 + \psi(x))}{1 - C\varepsilon^2} - \psi(x).$$

Then, by setting  $B_{k+1} = \frac{B_k + C(1 + \|\psi\|_{L^\infty(\overline{\Omega})})\varepsilon^2}{1 - C\varepsilon^2}$  we obtain

$$v^\varepsilon(x, t_k - \varepsilon^2) \geq -B_{k+1} - \psi(x).$$

As it is clear that  $B_{k+1} \leq B_k \frac{1 + C(1 + \|\psi\|_{L^\infty(\overline{\Omega})})\varepsilon^2}{1 - C\varepsilon^2}$ , we deduce that  $v^\varepsilon(x, T - k\varepsilon^2) \geq \tilde{B}_k - \psi(x)$  for all  $k$  with

$$\tilde{B}_k = B_0 \left( \frac{1 + C(1 + \|\psi\|_{L^\infty(\overline{\Omega})})\varepsilon^2}{1 - C\varepsilon^2} \right)^k.$$

Since  $k = (T - t)/\varepsilon^2$  and recalling that  $B_0 = B + \|\psi\|_{L^\infty(\overline{\Omega})}$ , we have shown that

$$v_\varepsilon(x, t) \geq -(B + \|\psi\|_{L^\infty(\overline{\Omega})})s_\varepsilon^{T-t} - \psi(x)$$

with

$$s_\varepsilon = \left( \frac{1 + C(1 + \|\psi\|_{L^\infty(\overline{\Omega})})\varepsilon^2}{1 - C\varepsilon^2} \right)^{1/\varepsilon^2}.$$

Since  $s_\varepsilon$  has a finite limit as  $\varepsilon \rightarrow 0$  we obtain a bound on  $v^\varepsilon$  of the desired form.  $\square$

**Remark 1.5.2.** By following the construction of the elliptic game presented in Section 1.2.3, we can take  $\psi = (\|h\|_{L^\infty} + 1)\psi_1$  where  $\psi_1$  is defined by (1.2.66). In that case,  $\|D\psi\|_{C_b^1(\overline{\Omega})} = \|D\psi_1\|_{C_b^1(\overline{\Omega})}(1 + \|h\|_{L^\infty})$ . This expression can be compared for a  $C^{2,\alpha}$ -domain to the estimate given by Remark 1.2.8 provided by the Schauder theory for which  $\|D\psi_1\|_{C_b^1(\overline{\Omega})}$  plays the role of the constant  $C_\Omega$  depending only on the domain.

## 1.5.2 The elliptic case

We shall assume throughout this section that the parameters  $M$  and  $m$  controlling the termination of the game are related by  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ ; in addition, we need to assume  $M$  is sufficiently large. Our plan is to show, using a fixed point argument, the existence of a function  $U^\varepsilon(x, z)$  (defined for all  $x \in \overline{\Omega}$  and  $|z| < M$ ) satisfying (3.3.2) and also

$$-z - \chi(x) \leq U^\varepsilon(x, z) \leq -z + \chi(x). \quad (1.5.1)$$

This implies that  $U^\varepsilon(x, z) < 0$  when  $z > \chi(x)$ , and  $U^\varepsilon(x, z) > 0$  when  $z < -\chi(x)$ . Recalling the definitions of  $u^\varepsilon$  and  $v^\varepsilon$ , it follows from (3.3.3)–(3.3.4) that

$$|v^\varepsilon(x)| \leq \chi(x), \quad |u^\varepsilon(x)| \leq \chi(x), \quad (1.5.2)$$

for all  $x \in \overline{\Omega}$ . It is convenient to work with  $V^\varepsilon(x, z) = U^\varepsilon(x, z) + z$  rather than  $U^\varepsilon$ , since this turns (1.5.1) into

$$|V^\varepsilon(x, z)| \leq \chi(x),$$

whose right-hand side is not constant. The dynamic programming principle (3.3.2) for  $U^\varepsilon$  is equivalent (after a bit of manipulation) to the statement that  $V^\varepsilon$  is a fixed point of the mapping  $\phi(\cdot, \cdot) \mapsto R_\varepsilon[\cdot, \cdot, \phi]$  where the operator  $R_\varepsilon$  is defined for any  $L^\infty$ -function  $\phi$  defined on  $\overline{\Omega} \times (-M, M)$  by

$$R_\varepsilon[x, z, \phi] = \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda \varepsilon^2} \phi(x', z') - \delta, & \text{if } |z'| < M, \\ -\chi(x), & \text{if } z' \geq M, \\ \chi(x), & \text{if } z' \leq -M. \end{cases} \quad (1.5.3)$$

where  $x' = x + \Delta x$  and  $z' = e^{\lambda \varepsilon^2}(z + \delta)$ , with  $\delta$  defined as in (3.3.1). Here  $p$ ,  $\Gamma$  and  $\Delta \hat{x}$  are constrained as usual by (1.2.16)–(1.2.17).

We shall identify  $V^\varepsilon$  as the unique fixed point of the mapping  $\phi(\cdot, \cdot) \mapsto R_\varepsilon[\cdot, \cdot, \phi]$  in  $F_\chi$  defined by

$$F_\chi = \{ \phi \in L^\infty(\overline{\Omega} \times (-M, M)) : \forall (x, z) \in \overline{\Omega} \times (-M, M), |\phi(x, z)| \leq \chi(x) \}. \quad (1.5.4)$$

**Lemma 1.5.3.** *Let  $\mathcal{F}$  satisfy (1.1.4) and (1.1.8)–(1.1.10) and assume as always that  $\alpha$ ,  $\beta$ ,  $\gamma$  fulfill (1.2.13)–(1.2.15) and that  $\Omega$  is a  $C^2$ -domain satisfying both the uniform interior and exterior ball conditions. Then, there exists  $M_0 > 0$  such that for all two positive constants  $m$  and  $M > M_0$  satisfying  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})} = M - 1$ , for any  $|z| \leq M$  and any  $x \in \overline{\Omega}$ , we have*

$$Q_\varepsilon[x, z, \chi] \leq \chi(x) \quad \text{and} \quad Q_\varepsilon[x, z, -\chi] \geq -\chi(x).$$

*Proof.* We are going to establish the upper estimate for  $\chi$ . By Lemma 1.4.19, we deduce that

$$Q_\varepsilon[x, z, \chi] - \chi(x) \leq \varepsilon^2 (1 + (\lambda - \eta)|z| + C_*) - \lambda \varepsilon^2 (m + \|\psi\|_{L^\infty(\overline{\Omega})} + \psi(x)).$$

Since  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})} = M - 1$  and  $|z| \leq M$ , we compute

$$Q_\varepsilon[x, z, \chi] - \chi(x) \leq \varepsilon^2 (1 + (\lambda - \eta)M + C_*) - \lambda \varepsilon^2 (M - 1 - \|\psi\|_{L^\infty(\overline{\Omega})} + \psi(x)).$$

By rearranging the terms, we obtain that

$$Q_\varepsilon[x, z, \chi] - \chi(x) \leq \varepsilon^2 \left( 1 + \lambda(1 + 2\|\psi\|_{L^\infty(\overline{\Omega})}) + C_* - \eta M \right).$$

We can choose  $M$  large enough such that the right-hand side is negative. It suffices to take

$$M > M_0 := \frac{1}{\eta} \left( 1 + \lambda(1 + 2\|\psi\|_{L^\infty(\overline{\Omega})}) + C_* \right).$$

The case for  $Q_\varepsilon[x, z, -\chi] \geq -\chi(x)$  is analogous. □

**Proposition 1.5.4.** *Assume the hypotheses of Lemma 1.5.3 hold. Suppose further that  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ . Then for all sufficiently small  $\varepsilon$ , the map  $\phi(\cdot, \cdot) \mapsto R_\varepsilon[\cdot, \cdot, \phi]$  is a contraction in the  $L^\infty$ -norm, which preserves  $F_\chi$ . In particular, it has a unique fixed point, which has  $L^\infty$ -norm at most  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})}$ .*

*Proof.* By the arguments already used in [88, Proposition 5.2], the map is a contraction for any  $\varepsilon$  (this part of the proof works for any  $M$ ). More precisely, if  $\phi_i$ ,  $i = 1, 2$  are two  $L^\infty$ -functions defined on  $\overline{\Omega} \times (-M, M)$  to  $\mathbb{R}$ , then  $\|R_\varepsilon[\cdot, \cdot, \phi_1] - R_\varepsilon[\cdot, \cdot, \phi_2]\|_{L^\infty} \leq e^{-\lambda\varepsilon^2} \|\phi_1 - \phi_2\|_{L^\infty}$ .

Now we prove that if  $M$  is large enough and  $m + 2\|\psi\|_{L^\infty(\overline{\Omega})} = M - 1$ , the map preserves the ball  $F_\chi$  defined by (1.5.4). Since  $R_\varepsilon[x, z, \phi]$  is monotone in its last argument, it suffices to show that

$$R_\varepsilon[x, z, \chi] \leq \chi(x) \quad \text{and} \quad R_\varepsilon[x, z, -\chi] \geq -\chi(x). \quad (1.5.5)$$

For the first inequality of (1.5.5), let  $p$  and  $\Gamma$  be fixed, and consider

$$\inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda\varepsilon^2} \chi(x') - \delta, & \text{if } |z'| < M, \\ -\chi(x), & \text{if } z' \geq M, \\ \chi(x), & \text{if } z' \leq -M. \end{cases} \quad (1.5.6)$$

If a minimizing sequence uses the second or third alternative then the inf is less than  $\chi(x)$ . In the remaining case, when all minimizing sequences use the first alternative, we apply Lemma 1.5.3 to see that (1.5.6) is bounded above by  $\chi(x)$ . It follows that for all  $x \in \overline{\Omega}$  and  $z \in (-M, M)$ ,  $R_\varepsilon[x, z, \chi] \leq \chi(x)$ , as asserted.

For the second inequality of (1.5.5), the argument is strictly parallel by considering the function  $-\chi$ . We have shown that the map  $\phi(\cdot, \cdot) \mapsto R_\varepsilon[\cdot, \cdot, \phi]$  preserves the ball  $F_\chi$ . Since it is also a contraction, the map has a unique fixed point.  $\square$

This result justifies the discussion of the stationary case given in Section 1.2, by showing that the value functions  $u^\varepsilon$  and  $v^\varepsilon$  are well-defined, and bounded independently of  $\varepsilon$ , and they satisfy the dynamic programming inequalities:

**Proposition 1.5.5.** *Suppose  $\mathcal{F}$  satisfies (1.1.4) and (1.1.8)–(1.1.10) the  $C^2$ -domain  $\Omega$  fulfills both the uniform interior and exterior ball conditions, and the boundary condition  $h$  is continuous, uniformly bounded. Assume the parameters of the game  $\alpha, \beta, \gamma$  fulfill (1.2.13)–(1.2.15),  $\psi \in C_b^2(\overline{\Omega})$  satisfy (1.2.55),  $M$  large enough,  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ , and  $\chi \in C_b^2(\overline{\Omega})$  is defined by (1.2.56). Let  $V^\varepsilon$  be the solution of (1.5.3) obtained by Proposition 1.5.4 and let  $U^\varepsilon(x, z) = V^\varepsilon(x, z) - z$ . Then the associated functions  $u^\varepsilon$ ,  $v^\varepsilon$  defined by (3.3.3)–(3.3.4) satisfy  $|u^\varepsilon| \leq \chi$  and  $|v^\varepsilon| \leq \chi$  for all sufficiently small  $\varepsilon$ , and they satisfy the dynamic programming inequalities (1.2.61) and (1.2.62) at all points  $x \in \overline{\Omega}$ .*

*Proof.* The bounds on  $u^\varepsilon$  and  $v^\varepsilon$  were demonstrated in (1.5.2). The bounds assure that the dynamic programming inequalities hold for all  $x \in \overline{\Omega}$ , as a consequence of Proposition 3.3.1.  $\square$

We close this section with the stationary analogue of Lemma 1.2.5.

**Lemma 1.5.6.** *Under the hypotheses of Proposition 1.5.3, suppose in addition that*

$$\mathcal{F}(x, z_1, p, \Gamma) \geq \mathcal{F}(x, z_0, p, \Gamma) \quad \text{whenever } z_1 > z_0.$$

*Then  $U^\varepsilon$  satisfies*

$$U^\varepsilon(x, z_1) \leq U^\varepsilon(x, z_0) - (z_1 - z_0) \quad \text{whenever } z_1 > z_0.$$

*In particular,  $U^\varepsilon$  is strictly decreasing in  $z$  and  $v^\varepsilon = u^\varepsilon$ .*

*Proof.* The Dirichlet case is provided in [88, Lemma 5.4]. For our game, it suffices to add  $-\|\Delta \hat{x} - \Delta x\| h(x + \Delta x)$  in the expression of  $\delta_0$  and  $\delta_1$  defined in the proof of [88, Lemma 5.4]. Then the arguments can be repeated on the operator  $R_\varepsilon$  defined by (1.5.3), noticing that the function  $\chi$  is independent of  $z$ .  $\square$

## 1.6 Some natural generalizations

In the precedent sections, we solved the Neumann boundary problem in both parabolic and elliptic settings. In the present section, we are going to explain without full proof how the previous work can be used to solve on the one hand the mixed Dirichlet-Neumann boundary conditions in the elliptic framework and on the other hand the oblique problem in the parabolic setting. For the definitions of the viscosity solutions on these frameworks which are the natural extensions of those presented in Section 1.3.1, the interested reader is referred to [11] or [45].

### 1.6.1 Elliptic PDE with mixed Dirichlet-Neumann boundary conditions

We extend the games of Section 1.2.3 devoted to the single Neumann problem to the mixed Dirichlet-Neumann boundary-value problem

$$\begin{cases} \mathcal{F}(x, u, Du, D^2u) + \lambda u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \Upsilon_D, \\ \frac{\partial u}{\partial n} = h, & \text{on } \Upsilon_N, \end{cases} \quad (1.6.1)$$

where  $\Omega \subsetneq \mathbb{R}^d$  is a domain,  $\Upsilon_D \cup \Upsilon_N = \partial\Omega$  is a partition of  $\partial\Omega$  with  $\Upsilon_D$  nonempty and closed and  $\Upsilon_N$  is assumed to be  $C^2$ . Then,  $\Omega$  is assumed to satisfy the uniform exterior ball condition and, in a neighborhood of  $\Upsilon_N$ , the uniform interior ball condition explained in Definition 1.1.2. We will need a  $C_b^2(\overline{\Omega})$ -function  $\psi$  such that

$$\frac{\partial \psi}{\partial n} = \|h\|_{L^\infty} + 1 \quad \text{on } \Upsilon_N. \quad (1.6.2)$$

From  $m$  and  $\psi$ , we construct a function  $\chi$  defined by

$$\chi(x) = m + \|\psi\|_{L^\infty} + \psi(x). \quad (1.6.3)$$

As in Section 1.2.3, we introduce  $U^\varepsilon(x, z)$ , the optimal worst-case present value of Helen's wealth if the initial stock is  $x$  and her initial wealth is  $-z$ . The definition of  $U^\varepsilon(x, z)$  for  $x \in \Omega \cup \Upsilon_N$  involves here a game similar to that of Section 1.2.3. The rules are as follows:

1. Initially, at time  $t_0 = 0$ , the stock price is  $x_0 = x$  and Helen's debt is  $z_0 = z$ .
2. Suppose, at time  $t_j = j\varepsilon^2$ , the stock price is  $x_j$  and Helen's debt is  $z_j$  with  $|z_j| < M$ . Then Helen chooses  $p_j \in \mathbb{R}^d$  and  $\Gamma_j \in \mathcal{S}_d$ , restricted in magnitude by (1.2.16). Knowing these choices, Mark determines the next stock price  $x_{j+1} = x_j + \Delta x$  so as to degrade Helen's outcome. Mark chooses an intermediate point  $\hat{x}_{j+1} = x_j + \Delta \hat{x}_j \in \mathbb{R}^d$  such that  $\|\Delta \hat{x}_j\| \leq \varepsilon^{1-\alpha}$ . This position  $\hat{x}_{j+1}$  determines the new position  $x_{j+1} = x_j + \Delta x_j$  by

$$x_{j+1} = \text{proj}_{\overline{\Omega}}(\hat{x}_{j+1}) \in \overline{\Omega}.$$

Helen experiences a loss at time  $t_j$  of

$$\delta_j = p_j \cdot \Delta \hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta \hat{x}_j, \Delta \hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(x_j, z_j, p_j, \Gamma_j) - \|\Delta \hat{x}_j - \Delta x_j\| h(x_j + \Delta x_j). \quad (1.6.4)$$

As a consequence, her time  $t_{j+1} = t_j + \varepsilon^2$  debt becomes  $z_{j+1} = e^{\lambda \varepsilon^2} (z_j + \delta_j)$ .

3. If  $z_{j+1} \geq M$ , then the game terminates, and Helen pays a "termination-by-large-debt penalty" worth  $e^{\lambda \varepsilon^2} (\chi(x_j) - \delta_j)$  at time  $t_{j+1}$ . Similarly, if  $z_{j+1} \leq -M$ , then the game terminates, and Helen receives a "termination-by-large-wealth bonus" worth  $e^{\lambda \varepsilon^2} (\chi(x_j) + \delta_j)$  at time  $t_{j+1}$ . If the game ends this way, we call  $t_{j+1}$  the "ending index"  $t_K$ .
4. If  $|z_{j+1}| < M$  and  $x_{j+1} \in \Upsilon_D$ , then the game terminates, and Helen gets an "exit payoff" worth  $g(x_{j+1})$  at time  $t_{j+1}$ . If the game ends this way, we call  $t_{j+1}$  the "exit index"  $t_E$ .

5. If the game has not terminated then Helen and Mark repeat this procedure at time  $t_{j+1} = t_j + \varepsilon^2$ .  
 If the game never stops, the “ending index”  $t_K$  is  $+\infty$ .

All the possibilities, apart the end by exit, had already been investigated at Section 1.2.3. If the game ends by exit at time  $t_E$ , then the present value of her income is

$$\begin{aligned} U^\varepsilon(x_0, z_0) &= -z_0 - \delta_0 - e^{-\lambda\varepsilon^2}\delta_1 - \dots - e^{-\lambda(E-1)\varepsilon^2}\delta_{E-1} + e^{-\lambda E\varepsilon^2}g(x_E) \\ &= e^{-\lambda E\varepsilon^2}(g(x_E) - z_E). \end{aligned}$$

Since the game is stationary, the associated dynamic programming principle is that for  $|z| < M$ ,

$$U^\varepsilon(x, z) = \sup_{p, \Gamma} \min_{\Delta \hat{x}} \begin{cases} e^{-\lambda\varepsilon^2}U^\varepsilon(x', z'), & \text{if } x' \in \Omega \cup \Gamma_N \text{ and } |z'| < M, \\ e^{-\lambda\varepsilon^2}(g(x') - z'), & \text{if } x' \in \Gamma_D \text{ and } |z'| < M, \\ -z - \chi(x), & \text{if } z' \geq M, \\ -z + \chi(x), & \text{if } z' \leq -M, \end{cases} \quad (1.6.5)$$

where  $x' = \text{proj}_{\overline{\Omega}}(x + \Delta \hat{x})$  and  $z' = e^{\lambda\varepsilon^2}(z + \delta)$ , with  $\delta$  defined by (1.6.4). Here  $p$ ,  $\Gamma$  and  $\Delta \hat{x}$  are constrained as usual by (1.2.16)–(1.2.17).

The definitions (3.3.3)–(3.3.4) of  $u^\varepsilon$  and  $v^\varepsilon$  on  $\Omega \cup \Gamma_N$  are conserved. The corresponding semi-relaxed limits are defined for any  $x \in \overline{\Omega}$  by

$$\overline{u}(x) = \limsup_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} u^\varepsilon(y) \quad \text{and} \quad \underline{v}(x) = \liminf_{\substack{y \rightarrow x \\ \varepsilon \rightarrow 0}} v^\varepsilon(y),$$

with the convention that  $y$  approaches  $x$  from  $\Omega \cup \Gamma_N$  (since  $u^\varepsilon$  and  $v^\varepsilon$  are only defined on  $\Omega \cup \Gamma_N$ ). Proposition 3.3.1 still holds without any modification for mixed-type Dirichlet-Neumann boundary conditions. Moreover, the definition of viscosity subsolutions and supersolutions is clear by relaxing the PDE condition on  $\Upsilon_D$  with the Dirichlet condition in the same way that has been done in [88, Section 3].

Following the same steps as our proof for the Neumann problem (the main modification consists in the proof of convergence on  $\Upsilon_D$  but has already been done in [88]), the following theorem is now immediate.

**Theorem 1.6.1.** *Consider the stationary boundary value problem (1.6.1) where  $\mathcal{F}$  satisfies (1.1.4) and (1.1.8)–(1.1.10),  $g$  and  $h$  are continuous, uniformly bounded and  $\Omega$  is a  $C^2$ -domain satisfying the uniform exterior ball condition and the uniform interior ball condition in a neighborhood of  $\Upsilon_N$ . Assume the parameters of the game  $\alpha, \beta, \gamma$  fulfill (1.2.13)–(1.2.15),  $\psi \in C_b^2(\overline{\Omega})$  satisfies (1.6.2),  $\chi \in C^2(\overline{\Omega})$  is defined by (1.6.3),  $M$  is sufficiently large, and  $m = M - 1 - 2\|\psi\|_{L^\infty(\overline{\Omega})}$ . Then  $u^\varepsilon$  and  $v^\varepsilon$  are well-defined when  $\varepsilon$  is sufficiently small, and they satisfy  $|u^\varepsilon| \leq \chi$  and  $|v^\varepsilon| \leq \chi$ . Their relaxed semi-limits  $\overline{u}$  and  $\underline{v}$  are respectively a viscosity subsolution and a viscosity supersolution of (1.6.1). If in addition we have  $\underline{v} \leq \overline{u}$  and the PDE has a comparison principle, then it follows that  $u^\varepsilon$  and  $v^\varepsilon$  converge locally uniformly in  $\overline{\Omega}$  to the unique viscosity solution of (1.6.1).*

## 1.6.2 Parabolic PDE with an oblique boundary condition

The target of this section is to construct a game which could interpret the PDE with an oblique condition  $h$  and final-time data  $g$  given by

$$\begin{cases} \partial_t u - \mathcal{F}(t, x, u, Du, D^2 u) = 0, & \text{for } x \in \Omega \text{ and } t < T, \\ \frac{\partial u}{\partial \varsigma}(x, t) = h(x), & \text{for } x \in \partial\Omega \text{ and } t < T, \\ u(x, T) = g(x), & \text{for } x \in \overline{\Omega}, \end{cases} \quad (1.6.6)$$

where  $\varsigma$  defines a smooth vector field, say  $C^2$ , on  $\partial\Omega$  pointing outward such that

$$\langle \varsigma(x), n(x) \rangle \geq \theta > 0 \quad \text{for all } x \in \partial\Omega. \quad (1.6.7)$$

As usual, the domain  $\Omega$  is supposed to be at least of boundary  $C^2$  and to satisfy both the uniform and the exterior ball conditions.

First of all, following P.L. Lions [106, Section 5], P.L. Lions and A.S. Sznitman [108], we introduce some smooth functions  $a_{ij}(x) = a_{ji}(x)$ , say  $C_b^2(\mathbb{R}^d)$ , such that

$$\begin{aligned} \exists \theta > 0, \forall x \in \mathbb{R}^d, (a_{ij}(x)) &\geq \theta I_d, \\ \forall x \in \partial\Omega, \sum_{j=1}^d a_{ij}(x) \varsigma_j(x) &= n_i(x) \quad \text{for } 1 \leq i \leq d. \end{aligned} \quad (1.6.8)$$

Clearly if we had  $\varsigma = n$ , we would just take  $a_{ij}(x) = \delta_{ij}$ . Next, the matrices induce a metric  $\text{dist}_\varsigma$  on  $\mathbb{R}^d$  defined by

$$\text{dist}_\varsigma(x, y) := \inf \left\{ \int_0^1 \left[ \sum_{1 \leq i, j \leq d} a_{ij}(\xi(t)) \dot{\xi}_i(t) \dot{\xi}_j(t) \right]^{1/2} dt : \xi \in C^1([0, 1], \mathbb{R}^d), \xi(0) = y, \xi(1) = x \right\}. \quad (1.6.9)$$

Then it is well known that for  $\|x - y\|$  small, there exists a unique minimizer in (1.6.9). The interested reader is referred to [106] for additional properties about  $\text{dist}_\varsigma$ . For this specific metric, we can now define for any  $x$  lying on a small  $\delta$ -neighborhood of the boundary a unique projection according the vector field  $\gamma$  along the boundary by

$$\bar{x}^\gamma = \text{proj}_{\bar{\Omega}}^\varsigma(x) \in \partial\Omega, \quad (1.6.10)$$

which corresponds to the unique minimum of  $\text{dist}_\varsigma(x, y)$  for  $y$  lying on the boundary. Finally,  $B_\varsigma(x, r)$  denotes the ball of center  $x$  and radius  $r$  induced by the metric  $\text{dist}_\varsigma$ .

We can now explain the rules of the game corresponding to the oblique problem (1.6.6). Let the parameters  $\alpha, \beta, \gamma$  satisfy (1.2.13)–(1.2.15). When the game begins, the position can have any value  $x_0 \in \bar{\Omega}$ ; Helen's initial score is  $y_0 = 0$ . The rules are as follows: if at time  $t_j = t_0 + j\varepsilon^2$  Helen's debt is  $z_j$  and the stock price is  $x_j$ , then

1. Helen chooses a vector  $p_j \in \mathbb{R}^d$  and a matrix  $\Gamma_j \in \mathcal{S}_d$ , restricted in magnitude by (1.2.16).
2. Taking Helen's choice into account, Mark chooses the stock price  $x_{j+1} = x_j + \Delta x_j$  so as to degrade Helen's outcome. Mark is going to choose an intermediate point  $\hat{x}_{j+1} = x_j + \Delta \hat{x}_j \in \mathbb{R}^d$  such that

$$\hat{x}_{j+1} \in B_\varsigma(x_j, \varepsilon^{1-\alpha}), \quad (1.6.11)$$

which determines the new position  $x_{j+1} = x_j + \Delta x_j \in \bar{\Omega}$  by the rule

$$x_{j+1} = \text{proj}_{\bar{\Omega}}^\varsigma(\hat{x}_{j+1}),$$

where  $\text{proj}_{\bar{\Omega}}^\varsigma$  is the projection defined by (1.6.10).

3. Helen's debt is changed to

$$z_{j+1} = z_j + p_j \cdot \Delta \hat{x}_j + \frac{1}{2} \langle \Gamma_j \Delta \hat{x}_j, \Delta \hat{x}_j \rangle + \varepsilon^2 \mathcal{F}(t_j, x_j, z_j, p_j, \Gamma_j) - d_\varsigma(\hat{x}_{j+1}, x_{j+1}) h(x_j + \Delta x_j).$$

4. The clock steps forward to  $t_{j+1} = t_j + \varepsilon^2$  and the process repeats, stopping when  $t_K = T$ . At the final time Helen receives  $g(x_K)$  from the option.

Rather than repeating the arguments already used, we are going to explain the modifications to carry out the analysis. First of all, by the boundedness of the  $a_{ij}$  and (1.6.8), the distance  $\text{dist}_\varsigma$  defined by (1.6.9) is equivalent to the euclidean distance. Since  $\Omega$  satisfies the uniform exterior ball condition, there exists, for a certain  $r_\varsigma > 0$ , a tubular neighborhood  $\{x \in \mathbb{R}^d \setminus \Omega : \text{dist}(x) < r_\varsigma\}$  of the boundary on which  $\text{proj}_{\bar{\Omega}}^\varsigma$  is well-defined. This guarantees the well-posedness of this game for all  $\varepsilon > 0$  small enough. Then,

if  $\text{dist}_\varsigma$  or the euclidean distance is used to compute  $D\phi$  and  $D^2\phi$  for a smooth function  $\phi$ , we will get the same results. Therefore, we can introduce the oblique analogues  $m_{\varsigma,\varepsilon}^x[\phi]$  and  $M_{\varsigma,\varepsilon}^x[\phi]$  of (1.3.1)–(1.3.2) by

$$m_{\varsigma,\varepsilon}^x[\phi] := \inf_{\substack{x+\Delta\hat{x}\notin\Omega \\ \Delta\hat{x}}} \{h(x+\Delta x) - D\phi(x) \cdot \varsigma(x+\Delta x)\}, \quad (1.6.12)$$

$$M_{\varsigma,\varepsilon}^x[\phi] := \sup_{\substack{x+\Delta\hat{x}\notin\Omega \\ \Delta\hat{x}}} \{h(x+\Delta x) - D\phi(x) \cdot \varsigma(x+\Delta x)\}, \quad (1.6.13)$$

where  $\Delta\hat{x}$  is constrained by (1.6.11) and  $\Delta x$  is determined by  $\Delta x = \text{proj}_\Omega^\varsigma(x + \Delta\hat{x}) - x$ . Thus, the particular choices  $p_{\text{opt}}^{m_\varsigma}$ ,  $p_{\text{opt}}^{M_\varsigma}$  and  $\Gamma_{\text{opt}}^\varsigma$  will be now respectively defined in the orthonormal basis  $\mathcal{B}_\varsigma = (e_1 = \varsigma(\bar{x}^\gamma), e_2, \dots, e_d)$  by

$$\begin{aligned} p_{\text{opt}}^{m_\varsigma}(x) &:= D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}_\varsigma(x)}{\varepsilon^{1-\alpha}} \right) m_{\varsigma,\varepsilon}^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}_\varsigma^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] \varsigma(\bar{x}^\gamma), \\ p_{\text{opt}}^{M_\varsigma}(x) &:= D\phi(x) + \left[ \frac{1}{2} \left( 1 - \frac{\text{dist}_\varsigma(x)}{\varepsilon^{1-\alpha}} \right) M_{\varsigma,\varepsilon}^x[\phi] - \frac{\varepsilon^{1-\alpha}}{4} \left( 1 - \frac{\text{dist}_\varsigma^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] \varsigma(\bar{x}^\gamma), \end{aligned}$$

and

$$\Gamma_{\text{opt}}^\varsigma(x) := D^2\phi(x) + \left[ \frac{1}{2} \left( -1 + \frac{\text{dist}_\varsigma^2(x)}{\varepsilon^{2-2\alpha}} \right) (D^2\phi(x))_{11} \right] E_{11},$$

where  $m_{\varsigma,\varepsilon}^x[\phi]$  and  $M_{\varsigma,\varepsilon}^x[\phi]$  are defined by (1.6.12)–(1.6.13), and  $E_{11}$  denotes the unit-matrix  $(1, 1)$  in the basis  $\mathcal{B}_\varsigma$ . The definitions of  $u^\varepsilon$ ,  $v^\varepsilon$  and their relaxed semi-limits  $\bar{u}$  and  $\underline{v}$ , given by (1.2.22)–(1.2.23) and (1.2.26), are conserved. The only change on the dynamic programming inequalities (1.2.24)–(1.2.25) concerning  $u^\varepsilon$  and  $v^\varepsilon$  is to replace  $\|\Delta\hat{x} - \Delta x\|$  by  $\text{dist}_\varsigma(x + \Delta\hat{x}, x + \Delta x)$ , and to constrain  $\Delta\hat{x}$  by (1.6.11). For stability, we need to consider a  $C_b^2(\bar{\Omega})$ -function  $\psi$  such that

$$\frac{\partial\psi}{\partial\varsigma}(x) = \|h\|_{L^\infty} + 1 \quad \text{on } \partial\Omega.$$

It is still allowed by the uniform interior ball condition applied to the  $C^2$ -domain  $\Omega$ . By using exactly the same ingredients already used for the Neumann problem and adapting the geometric estimates given by Section 1.4.1.1 in the oblique framework, we obtain the following theorem.

**Theorem 1.6.2.** *Consider the final-value problem (1.6.6) where  $\mathcal{F}$  satisfies (1.1.4)–(1.1.7),  $g$  and  $h$  are continuous, uniformly bounded,  $\Omega$  is a  $C^2$ -domain satisfying both the uniform interior and exterior ball conditions, and  $\varsigma$  is a continuous vector field on  $\partial\Omega$  and satisfy (1.6.7). Assume the parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  fulfill (1.2.13)–(1.2.15). Then  $\bar{u}$  and  $\underline{v}$  are uniformly bounded on  $\bar{\Omega} \times [t_*, T]$  for any  $t_* < T$ , and they are respectively a viscosity subsolution and a viscosity supersolution of (1.6.6). If the PDE has a comparison principle (for uniformly bounded solutions), then it follows that  $u^\varepsilon$  and  $v^\varepsilon$  converge locally uniformly to the unique viscosity solution of (1.6.6).*

**Acknowledgements:** I thank Sylvia Serfaty for bringing the problem to my attention and numerous helpful discussions. I thank Scott N. Armstrong for fruitful and encouraging talks and Guy Barles for helpful comments about viscosity solutions. Finally, I gratefully acknowledge support from the European Science Foundation through a EURYI award of Sylvia Serfaty.

# Quadratic expansions and partial regularity for fully nonlinear uniformly parabolic equations

Pour une équation parabolique associée à un opérateur uniformément elliptique, on obtient une estimée  $W^{3,\varepsilon}$ , qui fournit une borne inférieure sur la mesure de Lebesgue de l'ensemble sur lequel une solution de viscosité possède un développement quadratique. L'argument combine des estimations  $W^{2,\varepsilon}$  paraboliques avec un argument de principe de comparaison. En application, on montre, sous l'hypothèse supplémentaire que l'opérateur est  $C^1$ , que la solution de viscosité est de classe  $C^{2,\alpha}$  sur l'ensemble complémentaire d'un ensemble fermé dont la dimension de Hausdorff est  $\varepsilon$  strictement plus petite que celle de l'espace ambiant, où la constante  $\varepsilon > 0$  dépend seulement de la dimension et des constantes d'ellipticité.

Jean-Paul Daniel. Quadratic expansions and partial regularity for fully nonlinear uniformly parabolic equations. Article accepté dans <i>Calculus of Variations and Partial Differential Equations</i> . doi:10.1007/s00526-014-0783-0.
---





## Chapter 2

# Quadratic expansions and partial regularity for fully nonlinear uniformly parabolic equations

### Abstract

For a parabolic equation associated to a uniformly elliptic operator, we obtain a  $W^{3,\varepsilon}$  estimate, which provides a lower bound on the Lebesgue measure of the set on which a viscosity solution has a quadratic expansion. The argument combines parabolic  $W^{2,\varepsilon}$  estimates with a comparison principle argument. As an application, we show, assuming the operator is  $C^1$ , that a viscosity solution is  $C^{2,\alpha}$  on the complement of a closed set of Hausdorff dimension  $\varepsilon$  less than that of the ambient space, where the constant  $\varepsilon > 0$  depends only on the dimension and the ellipticity.

**Keywords:** partial regularity, fully nonlinear parabolic equations, viscosity solutions.

**Mathematics Subject Classification** 35B65, 35K55, 35D40, 49N60.

### Contents

<b>2.1</b>	<b>Introduction</b>	<b>107</b>
2.1.1	Motivation and statement	107
<b>2.2</b>	<b>Preliminaries and proof of the partial regularity result</b>	<b>109</b>
2.2.1	Hypotheses and notation	109
2.2.2	Preliminaries	110
2.2.3	Proof of Theorem 2.1.1	111
<b>2.3</b>	<b>Parabolic <math>W^{2,\varepsilon}</math> estimate</b>	<b>113</b>
<b>2.4</b>	<b>Proof of Theorem 2.1.2</b>	<b>130</b>

## 2.1 Introduction

### 2.1.1 Motivation and statement

In this paper, we prove a partial regularity result for viscosity solutions of the uniformly parabolic equation

$$\partial_t u + \mathcal{F}(D^2 u) = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^d \times (-1, \infty). \quad (2.1.1)$$

We write  $u$  as a function of  $(x, t) \in \mathbb{R}^d \times [-1, \infty)$  and  $D^2u$  denotes the Hessian of  $u$  with respect to the  $x$  variable. The operator  $\mathcal{F}$  is assumed to be uniformly elliptic and to have uniformly continuous first derivatives.

Recently, Armstrong, Silvestre and Smart [2] obtained a partial regularity result for viscosity solutions of the uniformly elliptic equation

$$\mathcal{F}(D^2u) = 0 \quad \text{in } U \subseteq \mathbb{R}^d, \quad (2.1.2)$$

with the same hypotheses on  $\mathcal{F}$ . Specifically, they proved that, for every  $0 < \alpha < 1$ , a viscosity solution of (2.1.2) is  $C^{2,\alpha}$  on the complement of a closed set of Hausdorff dimension strictly less than  $d$ .

We extend this result to the parabolic setting by showing that the singular set of a solution of (2.1.1) has Hausdorff dimension at most  $d + 1 - \varepsilon$ , where the constant  $\varepsilon > 0$  depends only on the ellipticity of  $\mathcal{F}$  and  $d$ . The hypotheses (F1) and (F2) are given in the next section. In this paper, Hölder spaces such as  $C^{2,\alpha}$  are to be understood in the parabolic sense (e.g., a parabolic  $C^{2,\alpha}$  function may only be  $C^{1,\alpha/2}$  in the time variable: see Section 2.1 for the precise definitions).

**Theorem 2.1.1.** *Assume that  $\mathcal{F}$  satisfies (F1) and (F2). Let  $u \in C(\Omega)$  be a viscosity solution of (2.1.1) in a domain  $\Omega \subseteq \mathbb{R}^{d+1}$ . Then there exist a constant  $\varepsilon > 0$ , depending only on  $d, \lambda, \Lambda$  and a closed subset  $\Sigma \subseteq \overline{\Omega}$  of Hausdorff dimension at most  $d + 1 - \varepsilon$ , such that, for every  $0 < \alpha < 1$ , the solution  $u$  belongs to  $C^{2,\alpha}(\Omega \setminus \Sigma)$ .*

A new difficulty arising in the proof of Theorem 2.1.1 in the parabolic setting is to obtain an analogue of the  $W^{3,\varepsilon}$  estimate, an important and useful tool from the regularity theory of nondivergence form elliptic equations [105, 34] (see also [2]). We prove it for viscosity solutions of the uniformly parabolic equation

$$\partial_t u + \mathcal{F}(D^2u) = g \quad \text{in } \Omega \subseteq \mathbb{R}^d \times (-1, \infty), \quad (2.1.3)$$

where  $g \in C^{0,1}(\Omega)$ . To give the precise statement of this result, we require some notation. We denote by  $\mathbb{M}_d$  the set of real  $d \times d$  matrices. The open ball of  $\mathbb{R}^d$  centered at  $x$  of radius  $\rho$  is denoted by  $B_\rho(x)$ . If  $x = 0$ , we simply write  $B_\rho$ . The following elementary cylindrical domains play a central role in the theory: for all  $\rho > 0$  and  $x \in \mathbb{R}^d$ , we define  $Q_\rho(x, t) := B_\rho(x) \times (t - \rho^2, t)$  and denote  $Q_\rho := Q_\rho(0, 0)$ .

Now we define, for  $u : \Omega \rightarrow \mathbb{R}$ , the quantity

$$\begin{aligned} \Psi(u, \Omega)(x, t) := & \inf \left\{ A \geq 0 : \exists (b, p, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } \forall (y, s) \in \Omega, s \leq t, \right. \\ & \left. |u(y, s) - u(x, t) - p \cdot (y - x) - b(s - t) - \frac{1}{2}(y - x) \cdot M(y - x)| \leq \frac{1}{6}A \left( |x - y|^3 + |s - t|^{3/2} \right) \right\}, \end{aligned}$$

which represents the cubic error in the best quadratic approximation of  $u$  at  $(x, t)$ . We emphasize this error is measured globally in  $\Omega \cap \{(y, s) : s \leq t\}$  and, of course, “cubic” in the time variable means cubic in  $t^{1/2}$ .

The statement of the parabolic  $W^{3,\varepsilon}$  estimate is given by the following theorem.

**Theorem 2.1.2** (Parabolic  $W^{3,\varepsilon}$  estimate). *Assume  $\mathcal{F}$  satisfies (F1),  $g \in C^{0,1}(Q_1)$  and  $u \in C(Q_1)$  solves (2.1.3) in  $Q_1$ . Then there exist universal constants  $C, \varepsilon > 0$  such that, for all  $\kappa > 0$ ,*

$$\left| \left\{ (x, t) \in Q_{1/2} \left( 0, -\frac{1}{4} \right) : \Psi(u, Q_{3/4})(x, t) > \kappa \right\} \right| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u| + |\mathcal{F}(0)| + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon}.$$

The elliptic analogue of Theorem 2.1.2 has been used for example to obtain quantitative estimates for the convergence of monotone finite difference schemes [35] as well as rates of convergence in homogenization [4, 36]. We expect Theorem 2.1.2 to have similar applications in the parabolic setting.

The overall idea of the proof of Theorem 2.1.2 is similar to the elliptic case: we differentiate the equation to obtain the result from the parabolic  $W^{2,\varepsilon}$  estimate. In the parabolic case, there is an extra

difficulty in controlling the derivative with respect to time. Unlike the elliptic case, here we need to use the PDE once more in order to show that by controlling all of the spatial derivatives we gain control over  $\partial_t u$ .

The argument for the partial regularity result is similar to the idea outlined in [2]. We apply a result of Wang [146], which asserts that any viscosity solution of (2.1.1) which is sufficiently close to a quadratic polynomial must be  $C^{2,\alpha}$ . This result is a generalization of a result of Savin [126] in the elliptic setting. Theorem 2.1.2 gives us such quadratic expansions except on a set of lower parabolic Hausdorff dimension.

**Structure of the article:** We start by gathering our notation and some preliminary results in Sections 2.2.1 and 2.2.2. The proof of Theorem 2.1.1 is presented in Section 2.2.3. Section 2.3 is devoted to the derivation of the parabolic  $W^{2,\varepsilon}$  estimate. Section 2.4 gives the proof of Theorem 2.1.2.

## 2.2 Preliminaries and proof of the partial regularity result

### 2.2.1 Hypotheses and notation

Let  $\mathbb{S}_d \subseteq \mathbb{M}_d$  be the set of symmetric matrices. If  $M \in \mathbb{M}_d$ ,  $M^\top$  denotes the transpose of  $M$ . Recall that the Pucci extremal operators are defined for constants  $0 < \lambda \leq \Lambda$  and  $M \in \mathbb{S}_d$  by

$$\mathcal{P}_{\lambda,\Lambda}^+(M) := \sup_{\lambda I_d \leq A \leq \Lambda I_d} -\text{tr}(AM), \quad \text{and} \quad \mathcal{P}_{\lambda,\Lambda}^-(M) := \inf_{\lambda I_d \leq A \leq \Lambda I_d} -\text{tr}(AM).$$

A convenient way to write the Pucci operators is

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = -\lambda \sum_{\mu_j > 0} \mu_j - \Lambda \sum_{\mu_j < 0} \mu_j \quad \text{and} \quad \mathcal{P}_{\lambda,\Lambda}^-(M) = -\Lambda \sum_{\mu_j > 0} \mu_j - \lambda \sum_{\mu_j < 0} \mu_j,$$

where  $\mu_1, \dots, \mu_d$  are the eigenvalues of  $M$ . These operators satisfy the inequalities

$$\mathcal{P}_{\lambda,\Lambda}^-(M) + \mathcal{P}_{\lambda,\Lambda}^-(N) \leq \mathcal{P}_{\lambda,\Lambda}^-(M + N) \leq \mathcal{P}_{\lambda,\Lambda}^-(M) + \mathcal{P}_{\lambda,\Lambda}^+(N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M + N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M) + \mathcal{P}_{\lambda,\Lambda}^+(N).$$

For the modern theory of nonlinear parabolic equations we refer to [92, 104]. A nice introduction to viscosity solutions of nonlinear parabolic equations can be found in [78]. Let  $Q \subseteq U \times (0, T)$  and  $\alpha \in (0, 1]$ . The set of upper and lower semicontinuous functions on  $Q$  are denoted by  $\text{USC}(Q)$  and  $\text{LSC}(Q)$ , respectively. We will use the following notation.

- $u \in C^{0,\alpha}(Q)$  means that there exists  $C > 0$  such that for all  $(x, t), (y, s) \in Q$ , we have

$$|u(x, t) - u(y, s)| \leq C(|x - y|^\alpha + |t - s|^{\frac{\alpha}{2}}).$$

In other words,  $u$  is  $\frac{\alpha}{2}$ -Hölder continuous in  $t$  and  $Du$  is  $\alpha$ -Hölder continuous in  $x$ .

- $u \in C^{1,\alpha}(Q)$  means that  $u$  is  $\frac{\alpha+1}{2}$ -Hölder continuous in  $t$  and  $Du$  is  $\alpha$ -Hölder continuous in  $x$ .
- $u \in C^{2,\alpha}(Q)$  means that  $\partial_t u$  is  $\frac{\alpha}{2}$ -Hölder continuous in  $t$  and  $D^2 u$  is  $\alpha$ -Hölder continuous in  $x$ .

Throughout this paper, the nonlinear elliptic operator  $\mathcal{F} : \mathbb{S}_d \rightarrow \mathbb{R}$  satisfies each of the two following conditions:

- (F1)  $\mathcal{F}$  is uniformly elliptic and Lipschitz; precisely, we assume that there exist constants  $0 < \lambda \leq \Lambda$  such that, for every  $M, N \in \mathbb{S}_d$ ,

$$\mathcal{P}_{\lambda,\Lambda}^-(M - N) \leq \mathcal{F}(M) - \mathcal{F}(N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M - N).$$

- (F2)  $\mathcal{F}$  is  $C^1$  and its derivative  $D\mathcal{F}$  is uniformly continuous, that is, there exists an increasing continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that  $\omega(0) = 0$  and for every  $M, N \in \mathbb{S}_d$ ,

$$|D\mathcal{F}(M) - D\mathcal{F}(N)| \leq \omega(|M - N|).$$

We call a constant *universal* if it depends only on the dimension  $d$  and the ellipticity constants  $\lambda$  and  $\Lambda$ . If  $U \subseteq \mathbb{R}^d$  and  $s < t$ , then the parabolic boundary of  $U \times (s, t)$  is denoted by

$$\partial_p(U \times (s, t)) := U \times \{s\} \cup \partial U \times (s, t).$$

It is convenient to work with the *parabolic Hausdorff dimension* of a set  $E \subseteq \mathbb{R}^{d+1}$ , which is defined by

$$\mathcal{H}_{\text{par}}(E) := \inf \left\{ 0 \leq s < +\infty : \text{for all } \delta > 0, \text{ there exists a collection } \{Q_{r_j}(x_j, t_j)\} \right. \\ \left. \text{of cylinders such that } E \subseteq \bigcup_{j=1}^{+\infty} Q_{r_j}(x_j, t_j) \text{ and } \sum_{j=1}^{+\infty} r_j^s < \delta \right\}.$$

The relationship between the parabolic Hausdorff measure  $\mathcal{H}_{\text{par}}(E)$  and the classical Hausdorff measure  $\mathcal{H}(E)$  is given by

$$2\mathcal{H}(E) - d \leq \mathcal{H}_{\text{par}}(E) \leq \mathcal{H}(E) + 1. \quad (2.2.1)$$

The reader is referred to [137] for further details about the parabolic framework. We remark that  $\mathcal{H}_{\text{par}}(\mathbb{R}^{d+1}) = d + 2$ .

### 2.2.2 Preliminaries

First we recall an interior  $C^{1,\alpha}$  regularity result for solutions of (2.1.3).

**Proposition 2.2.1** ([144, Section 4.2]). *If  $u$  is a viscosity solution of (2.1.3) in  $Q_1$ , then  $u \in C^{1,\alpha}(\overline{Q}_{1/2})$  for some universal  $0 < \alpha < 1$ . Moreover, there exists some universal constant  $C$  such that*

$$\sup_{\overline{Q}_{1/2}} |Du| \leq C \left( \sup_{Q_1} |u| + |\mathcal{F}(0)| + \|g\|_{C^{0,1}(Q_1)} \right).$$

Moreover, it is natural in the parabolic framework to introduce some other sets called parabolic balls. Precisely, given  $(x, t) \in \mathbb{R}^{d+1}$ , we define *parabolic balls* of opening  $\theta > 0$  and height  $h \geq 0$  to be closed subsets of the following form

$$G_{\theta,h}(x, t) := \{(y, s) \in \mathbb{R}^{d+1} : \theta|y - x|^2 \leq s - t \leq h\}$$

and

$$G_{\theta,h}^-(x, t) := \{(y, s) \in \mathbb{R}^{d+1} : \theta|y - x|^2 \leq t - s \leq h\}.$$

By direct computation,

$$|G_{\theta,h}(x, t)| = |G_{\theta,h}^-(x, t)| = \frac{2\omega_d}{d+2} h^{1+d/2} \theta^{-d/2}, \quad (2.2.2)$$

where  $\omega_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . We next collect some standard material about parabolic balls. The following lemma is a slight modification of [146, Lemma 2.2].

**Lemma 2.2.2** (Y. Wang, [146]). *Let  $\theta \geq 3/4$ . For all  $(x, t) \in G_{\theta,h_0}^-(x_0, t_0)$  and  $0 < h \leq t_0 - t$ , there exists a cylinder  $Q_r(x_2, t_2)$  such that*

$$t_2 = t + \frac{h}{2}, \quad r \geq \frac{1}{\nu} \sqrt{\frac{h}{\theta}} \quad \text{with} \quad \nu := \frac{4}{\sqrt{2}-1},$$

which satisfies the three following properties:

(P1)  $\overline{Q}_r(x_2, t_2) \subseteq G_{\theta,h}(x, t) \cap G_{\theta,h_0}^-(x_0, t_0) \cap \{(y, s) : t + h/4 \leq s \leq t + h/2\}$ .

(P2)  $|Q_r(x_2, t_2)|/|G_{\theta,h}(x, t)| \geq \eta_0$  where  $\eta_0$  depends only on  $d$  and  $\theta$ .

(P3) If  $(z, t) \in Q_{r/4}(x_2, t_2)$ , then for every  $(y, s) \in G_{1/2, r^2/16}^-(z, t - \frac{r^2}{16})$ ,

$$G_{1/2, t-s}(y, s) \subseteq Q_r(x_2, t_2).$$

*Proof.* The reader is referred to [146, Lemma 2.2] for (P1) and (P3). For (P2), we compute

$$|Q_r(x_2, t_2)| = r^{d+2} \geq \left(\frac{\sqrt{2}-1}{4}\right)^{d+2} \theta^{-d/2-1} h^{1+d/2} = \frac{d+2}{2\omega_d} \left(\frac{\sqrt{2}-1}{4}\right)^{d+2} \theta^{-1} |G_{\theta, h}(x, t)|.$$

It suffices to set  $\eta_0 := \frac{d+2}{2^{d+3}\omega_d} \left(\frac{\sqrt{2}-1}{4}\right)^{d+2} \theta^{-1}$  to get the desired estimate.  $\square$

Finally, we recall a Vitali-type covering lemma for parabolic balls which was already used and proved in [146] (and essentially follows from the standard argument for Vitali's covering lemma). It is a convenient alternative in the parabolic setting to the “stacked” estimate lemma and the Calderón-Zygmund decomposition (see [78]). Given  $(x, t) \in \mathbb{R}^{d+1}$  and  $h > 0$ , we define the parabolic ball  $\widehat{G}_{\theta, h}(x, t) \supseteq G_{\theta, h}(x, t)$  by

$$\widehat{G}_{\theta, h}(x, t) := G_{\hat{\theta}, 4h}(x, t - 3h) \quad \text{with} \quad \hat{\theta} := \frac{\theta}{(\sqrt{2} + 1)^2}.$$

We observe from (2.2.2) that the following ratio is a universal constant depending only on the dimension:

$$\eta := \frac{|G_{\theta, h}(x, t)|}{|\widehat{G}_{\theta, h}(x, t)|} = 4^{-(1+d/2)} (\sqrt{2} + 1)^{-d}. \quad (2.2.3)$$

**Lemma 2.2.3** (Vitali's lemma for parabolic balls). *Assume that  $E \subseteq \mathbb{R}^{d+1}$  is bounded and  $h : E \rightarrow \mathbb{R}$  is positive. Consider the following collection of parabolic balls:*

$$\left\{ G_{\theta, h(x, t)}(x, t) : (x, t) \in E \right\}.$$

*If  $\sup\{h(x, t) : (x, t) \in E\} < \infty$ , we can extract a countable subcollection  $\{G_{\theta, h(x_i, t_i)}(x_i, t_i) : i \in \mathbb{N}\}$  of disjoint parabolic balls such that*

$$E \subseteq \bigcup_{i \in \mathbb{N}} \widehat{G}_{\theta, h(x_i, t_i)}(x_i, t_i).$$

Finally, we give the statement of a proposition required to obtain the partial parabolic result. This proposition was obtained by Y. Wang [146] and is the parabolic analogue of a first result of Savin [126]. It gives  $C^{2, \alpha}$  regularity for flat viscosity solutions of uniformly parabolic equations. Roughly speaking, it states that a viscosity solution of a uniformly parabolic equation that is sufficiently close to a quadratic polynomial is, in fact, a classical solution.

**Proposition 2.2.4** (Y. Wang, [146]). *Suppose in addition to (F1)–(F2) that  $\mathcal{F}(0) = 0$ . Suppose that  $0 < \alpha < 1$  and  $u \in C(Q_1)$  is a solution of (2.1.1) in  $Q_1$ . Then there exists a constant  $\delta_0 > 0$  depending only on the ellipticity constants  $\lambda$  and  $\Lambda$ , the dimension  $d$ , the modulus of continuity  $\omega$ , and  $\alpha$ , such that*

$$\sup_{Q_1} |u| \leq \delta_0 \quad \text{implies that} \quad u \in C^{2, \alpha}(Q_{1/2}).$$

### 2.2.3 Proof of Theorem 2.1.1

The strategy of the proof is the following. By a covering argument, we can cover the singular set by parabolic balls centered in points for which  $\Psi$  presents large values. This is allowed by Lemma 2.2.5 that shows that  $u$  is not  $C^{2, \alpha}$  close to  $(x_0, t_0)$  implies  $\Psi(x_0, t_0)$  is large. Then the parabolic  $W^{3, \varepsilon}$  estimate given by Theorem 2.1.2 provides an upper bound on the size of the set of the bad points.

**Lemma 2.2.5.** *Suppose  $u \in C(Q_1)$  solves (2.1.1) in  $Q_1$  and  $0 < \alpha < 1$ . There is a universal constant  $\delta_\alpha > 0$ , depending on  $d, \lambda, \Lambda, \omega$  and  $\alpha$  such that for every  $(y, s_0) \in Q_{1/2}(0, -\frac{1}{4})$  and  $0 < r < 1/20$ ,*

$$\{\Psi(u, Q_{3/4}) \leq r^{-1}\delta_\alpha\} \cap Q_r(y, s_0) \neq \emptyset \quad \text{implies that} \quad u \in C^{2,\alpha}(Q_r(y, s_0 - r^2)).$$

*Proof.* Let  $\delta > 0$  to be adjusted. Suppose that  $0 < r < 1/20$ ,  $(y, s_0) \in Q_{1/2}(0, -\frac{1}{4})$  and  $(z, s) \in Q_r(y, s_0)$  is such that

$$\Psi(u, Q_{3/4})(z, s) \leq r^{-1}\delta.$$

Then there exist  $b \in \mathbb{R}$ ,  $p \in \mathbb{R}^d$  and  $M \in \mathbb{M}_d$  such that, for every  $(x, t) \in Q_{3/4}$  such that  $t \leq s$ ,

$$|u(x, t) - u(z, s) - p \cdot (x - z) - b(t - s) - \frac{1}{2}(x - z) \cdot M(x - z)| \leq \frac{1}{6}r^{-1}\delta(|x - z|^3 + |s - t|^{3/2}). \quad (2.2.4)$$

Replacing  $M$  by  $\frac{1}{2}(M + M^\top)$ , we may assume that  $M \in \mathbb{S}_d$ . Since  $u$  is a viscosity solution of (2.1.1), it is clear that

$$b + \mathcal{F}(M) = 0.$$

For  $(x, \tau) \in Q_1$ , define

$$v(x, \tau) := \frac{1}{16r^2} (u(z + 4rx, s + 16r^2\tau) - u(z, s) - 4rp \cdot x - 16br^2\tau - 8r^2x \cdot Mx).$$

Noticing that  $(z + 4rx, s + 16r^2\tau) \in Q_{3/4}$  for  $(x, \tau) \in Q_1$ , the inequality (2.2.4) implies that

$$\sup_{Q_1} |v(x, \tau)| \leq \frac{4}{3}\delta.$$

Define the operator  $\tilde{\mathcal{F}}(N) := b + \mathcal{F}(N + M)$ , and observe  $\tilde{\mathcal{F}}$  satisfies (F1) and (F2), with the same ellipticity constants  $\lambda, \Lambda$  and modulus  $\omega$ , and  $\tilde{\mathcal{F}}(0) = b + \mathcal{F}(M) = 0$ . It is clear that  $v$  is a solution of

$$\partial_t v + \tilde{\mathcal{F}}(D^2 v) = 0, \quad \text{in } Q_1.$$

Let  $\delta_0 > 0$  be the universal constant in Proposition 2.2.4, which also depends on  $\alpha$ . Suppose that

$$\delta < \frac{3}{4}\delta_0.$$

Then Proposition 2.2.4 yields that  $v \in C^{2,\alpha}(Q_{1/2})$ , from which we deduce that  $u \in C^{2,\alpha}(Q_{2r}(z, s))$ . Since  $Q_r(y, s_0 - r^2) \subseteq Q_{2r}(z, s)$ , the proof is complete.  $\square$

We now give the proof of the first main result.

*Proof of Theorem 2.1.1.* We assume without loss of generality that  $\mathcal{F}(0) = 0$ . By a standard covering argument, we may fix  $0 < \alpha < 1$  and assume that  $\Omega = Q_1$ ,  $u \in C(Q_1)$  is bounded, and it suffices to show that

$$u \in C^{2,\alpha}(V \setminus \Sigma) \quad \text{with} \quad V := B_{9/20} \times \left(-\frac{1}{2}, -\frac{1}{4} - \frac{3}{800}\right).$$

for a set  $\Sigma \subseteq \bar{V}$  with  $\mathcal{H}_{\text{par}}(\Sigma) \leq d + 2 - \varepsilon$ . Since, for every  $\beta > 0$ , the operator  $\mathcal{F}_\beta(M) := \beta^{-1}\mathcal{F}(\beta M)$  satisfies both (F1) and (F2) with the same constants  $\lambda$  and  $\Lambda$  but a different modulus  $\omega$  and the constant  $\varepsilon$  that we obtain does not depend on  $\omega$ , we may therefore assume without loss of generality that  $\sup_{Q_1} |u| \leq 1$ . Let  $\Sigma \subseteq V$  denote the set

$$\Sigma := \{(x, s) \in V : u \notin C^{2,\alpha}(Q_r(x, s + \frac{1}{2}r^2)) \text{ for every } r > 0\}.$$

Notice that  $\Sigma$  is closed, and thus compact. Fix  $0 < r < 1/20$ . According to the Vitali covering theorem for parabolic cylinders [104, Lemma 7.8], there exists a finite collection  $\{Q_r(x_i, s_i + \frac{1}{2}r^2)\}_{1 \leq i \leq m}$  of disjoint parabolic cylinders of radius  $r$ , with centers  $(x_i, s_i) \in \Sigma$ , such that

$$\Sigma \subseteq \bigcup_{i=1}^m Q_{5r}\left(x_i, s_i + \frac{25}{2}r^2\right).$$

Since  $(x_i, s_i) \in \Sigma$ , according to Lemma 2.2.5 there exists a constant  $\delta$  such that

$$\Psi(u, Q_{3/4})(y, \tau) > r^{-1}\delta \quad \text{for every } (y, \tau) \in \bigcup_{i=1}^m Q_r\left(x_i, s_i + \frac{3}{2}r^2\right).$$

Applying Theorem 2.1.2 to  $Q_r(x_i, s_i + \frac{3}{2}r^2) \subseteq Q_{1/2}(0, -\frac{1}{4})$ , we deduce that

$$mr^{d+2} = m \frac{|Q_r|}{|Q_1|} \leq C(r^{-1}\delta)^{-\varepsilon}$$

for some universal constants  $C, \varepsilon > 0$ . Therefore,

$$\sum_{i=1}^m (5r)^{d+2-\varepsilon} \leq 5^{d+2-\varepsilon} mr^{d+2-\varepsilon} \leq 5^{d+2-\varepsilon} C \delta^{-\varepsilon} < +\infty.$$

In particular, this implies that

$$\mathcal{H}_{\text{par}}(\Sigma) \leq d + 2 - \varepsilon.$$

By using (2.2.1), we get

$$\mathcal{H}(\Sigma) \leq \frac{1}{2}(d + d + 2 - \varepsilon) = d + 1 - \frac{\varepsilon}{2}. \quad \square$$

## 2.3 Parabolic $W^{2,\varepsilon}$ estimate

In this section, we state and prove the parabolic  $W^{2,\varepsilon}$  estimate associated to (2.1.3). It will be useful to prove Theorem 2.1.2 in Section 2.4. This result is essentially well-known but we give the argument here for the sake of completeness.

The key argument to prove Proposition 2.3.1 relies on a measure estimate on small parabolic balls stated in Lemma 2.3.6. Its proof consists in a suitable parabolic Aleksandrov-Bakelman-Pucci (ABP) inequality and a comparison principle achieved with a barrier function. Then we derive an induction relation in Lemma 2.3.5 by Lemma 2.2.3. Finally some classical arguments permit to obtain Proposition 2.3.1.

Let  $\Omega \subseteq \mathbb{R}^{d+1}$  and  $u, -v \in \text{LSC}(\Omega)$ . We say that  $v$  *touches  $u$  from below* at  $(x, t) \in \Omega$  if

$$\begin{cases} v(z, \tau) \leq u(z, \tau), & \text{for } (z, \tau) \in \Omega \text{ and } \tau \leq t, \\ v(x, t) = u(x, t). \end{cases}$$

We say that  $u$  *touches  $v$  from above* at  $(x, t) \in \Omega$  if  $v$  touches  $u$  at  $(x, t) \in \Omega$  from below. Let  $(y, s) \in \mathbb{R}^{d+1}$ . A polynomial  $P$  is called a *concave paraboloid* of opening  $\kappa > 0$  if

$$P_{y,s;\kappa}(z, \tau) = -\frac{\kappa}{2}|z - y|^2 + \kappa(\tau - s).$$

Similarly, a polynomial  $P$  is called a *convex paraboloid* of opening  $\kappa > 0$  if

$$P_{y,s;\kappa}(z, \tau) = \frac{\kappa}{2}|z - y|^2 - \kappa(\tau - s).$$



To state the estimate, we require some notation. Given a domain  $\Omega \subseteq \mathbb{R}^{d+1}$ , and a function  $u \in \text{LSC}(\Omega)$ , define the quantity

$$\underline{\Theta}(x, t) = \underline{\Theta}(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ s.t. } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. u(y, s) \geq u(x, t) + p \cdot (y - x) - A \left( \frac{1}{2} |x - y|^2 + (t - s) \right) \right\}.$$

Similarly, for  $u \in \text{USC}(\Omega)$ ,

$$\overline{\Theta}(x, t) = \overline{\Theta}(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ s.t. } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. u(y, s) \leq u(x, t) + p \cdot (y - x) + A \left( \frac{1}{2} |x - y|^2 + (t - s) \right) \right\},$$

and, for  $u \in C(\Omega)$ ,

$$\Theta(x, t) = \Theta(u, \Omega)(x, t) := \max \{ \underline{\Theta}(u, \Omega)(x, t), \overline{\Theta}(u, \Omega)(x, t) \}.$$

The quantity  $\underline{\Theta}(x, t)$  is the minimum curvature of any paraboloid that touches  $u$  from below at  $(x, t)$ . If  $u$  cannot be touched from below at  $(x, t)$  by any paraboloid, then  $\underline{\Theta}(x, t) = +\infty$ . A similar statement holds for  $\overline{\Theta}(x, t)$ , where we touch from above instead. Moreover, a function  $u$  is  $C^{1,1}$  on a closed set  $\Gamma \subseteq \Omega$  if and only if  $u$  has tangent paraboloids from above and below with respect to  $\Omega$  at each point of  $\Gamma$ .

The form of the  $W^{2,\varepsilon}$  estimate we need is given by the following proposition.

**Proposition 2.3.1.** *If  $u \in \text{LSC}(Q_1)$  and  $L \geq 0$  satisfy the inequality*

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -L \quad \text{in } Q_1,$$

*then for all  $\kappa > 0$ ,*

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u, Q_1)(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u| + L} \right)^{-\varepsilon},$$

*where the constants  $C$  and  $\varepsilon > 0$  are universal.*

We emphasize here that  $\underline{\Theta}(u, \Omega)$  is defined in terms of quadratic polynomials that touch  $u$  at  $(x, t)$  and stay below  $u$  in the domain  $\Omega \cap \{(y, s) : s \leq t\}$ , which is full in space and restricted to global times less than  $t$ .

Instead of working with the sets  $\{\underline{\Theta} \leq \kappa\}$ , we are going to consider some new sets  $A_\kappa$  for  $\kappa > 0$ . We are inspired from the elliptic definition introduced by Savin [126] and recently also used by Armstrong and Smart in [5]. In the parabolic setting, define, for every  $\kappa > 0$ ,

$$A_\kappa := \left\{ (x, t) \in Q_1 : \exists (y, s) \in B_1 \times (-1, t] \text{ s.t. } u(x, t) - \inf_{Q_1} u + \kappa \left( \frac{1}{2} |x - y|^2 - (t - s) \right) \right. \\ \left. = \inf_{\substack{(z, \tau) \in Q_1, \\ \tau \leq t}} \left( u(z, \tau) - \inf_{Q_1} u + \kappa \left( \frac{1}{2} |z - y|^2 - (\tau - s) \right) \right) = 0 \right\}. \quad (2.3.1)$$

It is important to notice that  $s$  takes part in the definition of  $A_\kappa$  only to adjust the infimum to be equal to zero. Moreover, the definition of  $A_\kappa$  given above is adapted to the domain  $Q_1$ . It is clear how to change this definition of  $A_\kappa$  to deal with more general domains.

The next lemma gathers some properties about the sets  $A_\kappa$ . In particular, the link between  $A_\kappa$  and  $\underline{\Theta}$  is precised.

**Lemma 2.3.2.** *Let  $u \in \text{LSC}(\Omega)$ ,  $\kappa > 0$  and  $A_\kappa$  be defined by (2.3.1). Then we have*

$$A_\kappa \subseteq \{(x, t) \in Q_1 : \underline{\Theta}(u, Q_1)(x, t) \leq \kappa\}.$$

*Moreover, for all  $0 < \kappa_1 \leq \kappa$ , we have  $A_{\kappa_1} \subseteq A_\kappa$ .*

*Proof.* Let  $(x, t) \in A_\kappa$ . Then  $(x, t) \in Q_1$  and there exists  $y \in B_1$  such that for all  $(z, \tau) \in Q_1$ ,  $\tau \leq t$ , we have

$$u(z, \tau) + \kappa \left( \frac{1}{2} |z - y|^2 - \tau \right) \geq u(x, t) + \kappa \left( \frac{1}{2} |x - y|^2 - t \right).$$

After rearranging the terms, we get

$$u(z, \tau) \geq u(x, t) + \kappa \left( \frac{1}{2} (|x - y|^2 - |z - y|^2) - (t - \tau) \right).$$

An algebraic manipulation yields  $|x - y|^2 - |z - y|^2 = -|x - z|^2 + 2(y - x) \cdot (z - x)$ . If we choose  $p = \kappa(y - x) \in \mathbb{R}^d$ , the last inequality can be written in the form

$$u(z, \tau) \geq u(x, t) + p \cdot (z - x) - \kappa \left( \frac{1}{2} |x - z|^2 + (t - \tau) \right),$$

for all  $(z, \tau) \in Q_1$ ,  $\tau \leq t$ , which gives precisely  $\underline{Q}(u, Q_1)(x, t) \leq \kappa$ .

For the second assertion, let  $(x_1, t_1) \in A_{\kappa_1}$  and  $\kappa > \kappa_1$ . Hence there exist  $y_1 \in B_1$  and  $-1 < s_1 \leq t_1$  such that

$$\begin{aligned} u(x_1, t_1) - \inf_{Q_1} u + \kappa_1 \left( \frac{1}{2} |x_1 - y_1|^2 - (t_1 - s_1) \right) \\ = \inf_{\substack{(z, \tau) \in Q_1, \\ \tau \leq t}} \left( u(z, \tau) - \inf_{Q_1} u + \kappa_1 \left( \frac{1}{2} |z - y_1|^2 - (\tau - s_1) \right) \right) = 0. \end{aligned} \quad (2.3.2)$$

For  $(y, s) \in \mathbb{R}^{d+1}$ , let  $P$  be the paraboloid given by

$$P(z, \tau) := \kappa \left( \frac{1}{2} |z - y|^2 - (\tau - s) \right) - \kappa_1 \left( \frac{1}{2} |z - y_1|^2 - (\tau - s_1) \right). \quad (2.3.3)$$

Assume that we have shown there exists  $(y, s) \in B_1 \times (-1, t_1]$  such that, for all  $(z, \tau) \in \mathbb{R}^{d+1}$ ,

$$P(z, \tau) = (\kappa - \kappa_1) \left( \frac{1}{2} |z - x_1|^2 - (\tau - t_1) \right). \quad (2.3.4)$$

For this particular choice, the decomposition

$$u(z, \tau) - \inf_{Q_1} u + \kappa \left( \frac{1}{2} |z - y|^2 - (\tau - s) \right) = u(z, \tau) - \inf_{Q_1} u + \kappa_1 \left( \frac{1}{2} |z - y_1|^2 - (\tau - s_1) \right) + P(z, \tau)$$

implies the result by using (2.3.2),  $P \geq 0$  on  $B_1 \times (-1, t_1]$  and  $P(x_1, t_1) = 0$  by (2.3.4). To obtain the assertion, it remains to show (2.3.4). By completing the square in (2.3.3), the polynomial  $P$  can be written in the form

$$P(z, \tau) = (\kappa - \kappa_1) \left( \frac{1}{2} |z - \bar{z}(y, s)|^2 - (\tau - \bar{\tau}(y, s)) \right),$$

with

$$\begin{cases} \bar{z}(y, s) := \frac{1}{\kappa - \kappa_1} (\kappa y - \kappa_1 y_1), \\ \bar{\tau}(y, s) := \frac{1}{\kappa - \kappa_1} \left( \kappa s - \kappa_1 s_1 + \frac{1}{2} \frac{\kappa \kappa_1}{\kappa - \kappa_1} |y - y_1|^2 \right). \end{cases}$$

We choose  $(y, s)$  to impose the condition  $(\bar{z}(y, s), \bar{\tau}(y, s)) = (x_1, t_1)$ . This leads to select

$$y = \left( 1 - \frac{\kappa_1}{\kappa} \right) x_1 + \frac{\kappa_1}{\kappa} y_1 \quad \text{and} \quad s = t_1 - \frac{\kappa_1}{\kappa} \left( t_1 - s_1 + \frac{1}{2} |x_1 - y_1|^2 \left( 1 - \frac{\kappa_1}{\kappa} \right) \right).$$

It is clear by convexity that  $y \in B_1$ . To show that  $s \in (s_1, t_1]$ , notice  $t_1 - s_1 - \frac{1}{2} |x_1 - y_1|^2 \geq 0$  by (2.3.2) and use  $\kappa > \kappa_1$  in the expression above.  $\square$

Now we recall the standard tool in the theory of viscosity solutions (see [45] for further details). We denote the infimal convolution of  $u \in \text{LSC}(Q_1)$  by

$$u_\varepsilon(x, t) = \inf_{(z, \tau) \in Q_1} \left( u(z, \tau) + \frac{2}{\varepsilon} (|z - x|^2 + (\tau - t)^2) \right).$$

Moreover, if  $f \in C(Q_1)$  and

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq f \quad \text{in } Q_1,$$

then there exist a sequence of functions  $f_\varepsilon \in C(Q_1)$  which converges locally uniformly to  $f$  respectively, as  $\varepsilon \rightarrow 0$ , such that  $u_\varepsilon$  satisfies

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq f_\varepsilon \quad \text{in } Q_{1-r_\varepsilon}(0, T_\varepsilon),$$

when  $r_\varepsilon \rightarrow 0$  and  $T_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The function  $u_\varepsilon$  is more regular than  $u$  and, in particular, is semiconcave. It is a good approximation to  $u$  in the sense that  $u_\varepsilon \rightarrow u$  locally uniformly in  $Q_1$  as  $\varepsilon \rightarrow 0$ . For us, the main utility of these approximations is the semiconcavity of  $u_\varepsilon$ . If  $u_\varepsilon$  can be touched from below by a smooth function  $\varphi$  at some point  $(y, s) \in Q_1$ , then  $u_\varepsilon$  is  $C^{1,1}$  at  $(y, s)$ , with norm depending only on  $\varepsilon$  and  $|D^2 \varphi(y, s)|$  and  $\partial_t \varphi(y, s)$ .

The following lemma is the form of the ABP inequality we are going to use.

**Lemma 2.3.3.** *Assume that  $L > 0$  and  $u \in \text{LSC}(Q_1)$  satisfy*

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -L \quad \text{in } Q_1.$$

*Suppose that  $a > 0$  and  $V \subseteq \mathbb{R}^{d+1}$  is compact such that, for each  $(y, s) \in V$ , there exists  $(x, t) \in Q_1$  such that*

$$u(x, t) - \inf_{Q_1} u + \frac{a}{2}|x - y|^2 - a(t - s) = \inf_{(z, \tau) \in Q_1, \tau \leq t} \left( u(z, \tau) - \inf_{Q_1} u + \frac{a}{2}|z - y|^2 - a(\tau - s) \right) = 0. \quad (2.3.5)$$

*Let  $W := \{(x, t) \in Q_1 : (2.3.5) \text{ holds for } u \text{ for some } (y, s) \in V\}$ . Then*

$$|V| \leq \frac{1}{\lambda^d} \left( 1 + \frac{L}{a} + \Lambda d \right)^{d+1} |W|. \quad (2.3.6)$$

*Proof.* The proof is divided into two steps.

**Step 1.** We make two reductions. First, by replacing  $u$  by  $u + \alpha(\frac{1}{2}|x|^2 - t)$  and  $L$  by  $L + C\alpha$  and letting  $\alpha \rightarrow 0$ , we may suppose that there exists  $\eta > 0$  such that for every  $(y, s) \in Q_1$ ,

$$\min_{(z, \tau) \in Q_1 \setminus Q_{1-\eta}, \tau \leq t} \left( u(z, \tau) + \frac{a}{2}|z - y|^2 - a\tau \right) > \inf_{(z, \tau) \in Q_{1-\eta}, \tau \leq t} \left( u(z, \tau) + \frac{a}{2}|z - y|^2 - a\tau \right). \quad (2.3.7)$$

Next we make a reduction to the case that  $u$  is semiconcave by an infimal convolution approximation. According to (2.3.7), for every sufficiently small  $\varepsilon > 0$ , there exist  $0 < r_\varepsilon < 1$  and  $-1 < T_\varepsilon < 0$  such that, for each  $(y, s) \in V$ , there exists  $(x, t) \in Q_{1-r_\varepsilon}(0, T_\varepsilon)$  such that

$$\begin{aligned} u_\varepsilon(x, t) - \inf_{Q_{1-r_\varepsilon}(0, T_\varepsilon)} u_\varepsilon + \frac{a}{2}|x - y|^2 - a(t - s) \\ = \inf_{(z, \tau) \in Q_{1-r_\varepsilon}(0, T_\varepsilon), \tau \leq t} \left( u(z, \tau) - \inf_{Q_{1-r_\varepsilon}(0, T_\varepsilon)} u_\varepsilon + \frac{a}{2}|z - y|^2 - a(\tau - s) \right) = 0, \end{aligned} \quad (2.3.8)$$

and

$$\lim_{\varepsilon \rightarrow 0} r_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} T_\varepsilon = 0.$$

Set

$$W_\varepsilon := \{(x, t) \in Q_{1-r_\varepsilon}(0, T_\varepsilon) : (2.3.8) \text{ holds for } u_\varepsilon \text{ for some } (y, s) \in V\}.$$

Assume that we have shown that

$$\limsup_{\varepsilon \rightarrow 0} W_\varepsilon \subseteq W, \quad \text{where} \quad \limsup_{\varepsilon \rightarrow 0} W_\varepsilon := \bigcap_{\varepsilon > 0} \bigcup_{0 < \delta \leq \varepsilon} W_\delta, \quad (2.3.9)$$

and for all  $\varepsilon > 0$ ,

$$|V| \leq \frac{1}{\lambda^d} \left(1 + \frac{L}{a} + \Lambda d\right)^{d+1} |W_\varepsilon|. \quad (2.3.10)$$

Then, since  $\sup_{0 < \varepsilon < 1/2} |W_\varepsilon| \leq |Q_1| < +\infty$ , the inequality  $\limsup_{\varepsilon \rightarrow 0} |W_\varepsilon| \leq |\limsup_{\varepsilon \rightarrow 0} W_\varepsilon|$  holds true and we have

$$|V| \stackrel{(2.3.10)}{\leq} \frac{1}{\lambda^d} \left(1 + \frac{L}{a} + \Lambda d\right)^{d+1} \left| \limsup_{\varepsilon \rightarrow 0} W_\varepsilon \right| \stackrel{(2.3.9)}{\leq} \frac{1}{\lambda^d} \left(1 + \frac{L}{a} + \Lambda d\right)^{d+1} |W|.$$

Thus we deduce (2.3.6).

To obtain the lemma, it remains to show the assertions (2.3.9) and (2.3.10).

For (2.3.9), let  $(x, t) \in \limsup_{\varepsilon \rightarrow 0} W_\varepsilon$ . Up to a subsequence, we can assume that  $(x, t) \in W_\varepsilon \cap Q_{1-r_\varepsilon}(0, T_\varepsilon)$  for all  $\varepsilon > 0$ . By the definition of  $W_\varepsilon$ , there exists  $(y_\varepsilon, s_\varepsilon) \in V$  such that

$$\begin{aligned} u_\varepsilon(x, t) - \inf_{Q_{1-r_\varepsilon}(0, T_\varepsilon)} u_\varepsilon + \frac{a}{2}|x - y_\varepsilon|^2 - a(t - s_\varepsilon) \\ = \inf_{(z, \tau) \in Q_{1-r_\varepsilon}(0, T_\varepsilon), \tau \leq t} \left( u_\varepsilon(z, \tau) - \inf_{Q_{1-r_\varepsilon}(0, T_\varepsilon)} u_\varepsilon + \frac{a}{2}|z - y_\varepsilon|^2 - a(\tau - s_\varepsilon) \right) = 0. \end{aligned} \quad (2.3.11)$$

Since  $V$  is compact, up to extracting a subsequence, there exists  $(y, s) \in V$  such that  $(y_\varepsilon, s_\varepsilon) \rightarrow (y, s)$  as  $\varepsilon \rightarrow 0$ . By convergence of  $u_\varepsilon$ , we deduce that  $u_\varepsilon(x, t) \xrightarrow{\varepsilon \rightarrow 0} u(x, t)$ . Moreover, since  $u_\varepsilon \geq u$  and  $u_\varepsilon \rightarrow u$  locally uniformly on  $Q_1$ , we have

$$\lim_{\varepsilon \rightarrow 0} \inf_{Q_{1-r_\varepsilon}(0, T_\varepsilon)} u_\varepsilon = \inf_{Q_1} u.$$

Letting  $\varepsilon \rightarrow 0$  in (2.3.11) yields  $(x, t) \in W$ . This completes the proof of (2.3.9), and therefore it remains to prove (2.3.10), that is, the statement of the lemma under the extra assumption that  $u$  is semiconcave.

**Step 2.** Assuming  $u$  is semiconcave, we give the proof of (2.3.6). Select a Lebesgue-measurable function  $\overline{Z} : V \rightarrow Q_1$  such that the map

$$(z, t) \mapsto u(z, t) - \inf_{Q_1} u + a \left( \frac{1}{2}|z - y|^2 - (t - s) \right)$$

attains its infimum in  $Q_1$  at  $(z, t) = \overline{Z}(y, s)$  and this infimum is equal to zero. For example, we may take  $\overline{Z}(y, s)$  to be the lexicographically least element of the (necessarily closed) set of infima. The function  $u$  is  $C^{1,1}$  on  $A := \overline{Z}(V)$  and  $\overline{Z}$  has a Lipschitz inverse  $\overline{Y} = (\overline{y}, \overline{s})$  given by

$$\begin{cases} \overline{y}(z, t) = z + \frac{1}{a} Du(z, t), \\ \overline{s}(z, t) = t - \frac{1}{a} u(z, t) - \frac{1}{2}|z - \overline{y}(z, t)|^2. \end{cases}$$

By Rademacher's theorem,  $\overline{Y}$  is differentiable almost everywhere on  $A$  for the  $d+1$ -dimensional Lebesgue measure. Then, by using the Lebesgue differentiation theorem, we see that  $u$  is twice differentiable in space and differentiable in time at almost every point of  $(z, t) \in A$  and, at such  $(z, t)$ , we have

$$D^2 u(z, t) \geq -aI_d \quad \text{and} \quad \partial_t u(z, t) \leq a.$$

Thus,

$$D\overline{y}(z, t) = I_d + \frac{1}{a} D^2 u(z, t) \geq 0$$

as well as

$$\begin{aligned} -\lambda \operatorname{tr}(D\overline{y}(z, t)) &= \mathcal{P}_{\lambda, \Lambda}^+(D\overline{y}(z, t)) = \mathcal{P}_{\lambda, \Lambda}^+ \left( I_d + \frac{1}{a} D^2 u(z, t) \right) \\ &\geq \frac{1}{a} \mathcal{P}_{\lambda, \Lambda}^+(D^2 u(z, t)) + \mathcal{P}_{\lambda, \Lambda}^-(I_d) \geq \frac{1}{a} (-L - \partial_t u(z, t)) - \Lambda d, \end{aligned}$$

and therefore

$$0 \leq D\bar{y}(z, t) \leq \frac{1}{\lambda} \left( 1 + \frac{L}{a} + \Lambda d \right).$$

Similarly  $D^2u(z, t) \geq -aI_d$  implies that

$$1 - \frac{1}{a} \partial_t u(z, t) \leq 1 - \frac{1}{a} \left( -L - \mathcal{P}_{\lambda, \Lambda}^+(-aI_d) \right) = 1 + \frac{L}{a} + \Lambda d.$$

An application of the area formula for Lipschitz functions gives

$$\begin{aligned} |V| &= \int_A |\det D\bar{Y}(z, t)| dz dt = \int_A \det \left( I + \frac{1}{a} D^2u(z, t) \right) \left( 1 - \frac{1}{a} \partial_t u(z, t) \right) dz dt \\ &\leq \frac{1}{\lambda^d} \left( 1 + \frac{L}{a} + \Lambda d \right)^d \left( 1 + \frac{L}{a} + \Lambda d \right) |A|, \end{aligned}$$

from which we obtain the lemma, using that  $A \subseteq W$ .  $\square$

In our analysis, an important role will be played by the functions  $\phi$  which we define by

$$\phi(x, t) := c(t + \tau)^{-b} \left( e^{-a \frac{|x|^2}{t+\tau}} - e^{-a\theta^{-1}} \right).$$

The parameters  $a, b$  and  $c$  will be adjusted with the uniform ellipticity constants and the opening  $\theta$  of the parabolic balls of the form  $G_{\theta, 1+\tau}(0, -\tau)$  with  $\tau > 0$ . More precisely, we will consider the choice given by

$$a = \max \left\{ \frac{1 + d\Lambda\theta}{2\lambda}, \theta \right\} \quad (2.3.12)$$

and

$$b = \max \left\{ \frac{2d\Lambda a + 1}{1 - e^{\frac{1}{2} - a\theta^{-1}}}, \frac{4\lambda a - 1}{e^{\frac{1}{2} - a\theta^{-1}}} \right\} \quad (2.3.13)$$

and

$$c = 2(1 + \tau)^{b+1} e^{a\theta^{-1}}. \quad (2.3.14)$$

We next show that, with this choice of parameters,  $\phi$  is a nonnegative subsolution in  $G_{\theta, 1+\tau}(0, -\tau)$  which vanishes on the lateral boundary of  $G_{\theta, 1+\tau}(0, -\tau)$  and is not too large initially. This plays the role of the “bump function” from the elliptic case [34, Lemma 4.1].

**Lemma 2.3.4.** *Let  $\tau > 0$ . For  $a, b$  and  $c$  given by (2.3.12)–(2.3.14), the function  $\phi$  satisfies*

$$\begin{cases} \partial_t \phi + \mathcal{P}_{\lambda, \Lambda}^+(D^2\phi) \leq -1, & \text{on } G_{\theta, 1+\tau}(0, -\tau) \cap \{(y, s) : s > 0\}, \\ \phi = 0, & \text{on } \partial_p G_{\theta, 1+\tau}(0, -\tau) \cap \{(y, s) : s > 0\}, \\ \phi > 0, & \text{on } G_{\theta, 1+\tau}(0, -\tau) \setminus \partial_p G_{\theta, 1+\tau}(0, -\tau), \\ 0 \leq \phi \leq \beta, & \text{on } G_{\theta, 1+\tau}(0, -\tau) \cap \{(y, s) : s = 0\}, \end{cases}$$

with  $\beta > 0$  given by

$$\beta := 2 \frac{(1 + \tau)^{b+1}}{\tau^b} (e^{a\theta^{-1}} - 1).$$

*Proof.* Let us introduce the variable  $\rho := |x|^2/(t + \tau)$  and the function  $\psi$  given by

$$\psi(\rho, t) := c(t + \tau)^{-(b+1)} e^{-a\rho}.$$

First, by inserting the value of  $c$  given by (2.3.14), observe that

$$\psi(\rho, t) \geq 2 \quad \text{on } G_{\theta, 1+\tau}(0, -\tau). \quad (2.3.15)$$

The two last properties are immediate to check. Thus we focus on the first assertion. The time derivative of  $\phi$  is given by

$$\partial_t \phi(x, t) = \psi(\rho, t) \left( -(1 - e^{a(\rho - \theta^{-1})})b + a\rho \right).$$

and the Hessian of  $\phi$  is given by

$$D^2 \phi(x, t) = 2a\psi(\rho, t) \left( (-1 + 2a\rho) \frac{x \otimes x}{|x|^2} - \left( I - \frac{x \otimes x}{|x|^2} \right) \right).$$

and has eigenvalues  $2a\psi(\rho, t) (-1 + 2a\rho)$  with multiplicity 1 and  $-2a\psi(\rho, t)$  with multiplicity  $d - 1$ . Hence

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(x, t)) = \psi(\rho, t) \begin{cases} 2a(d - 2a\rho)\Lambda, & \text{if } \rho \leq (2a)^{-1}, \\ 2a(-\lambda(-1 + 2a\rho) + (d - 1)\Lambda), & \text{if } \rho \geq (2a)^{-1}. \end{cases}$$

We distinguish two cases. If  $0 \leq \rho \leq (2a)^{-1}$ ,

$$\begin{aligned} \partial_t \phi(x, t) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(x, t)) &= \psi(\rho, t) \left( -(1 - e^{a(\rho - \theta^{-1})})b + a(\rho + 2(d - 2a\rho)\Lambda) \right) \\ &\stackrel{(2.3.12)}{\leq} \psi(\rho, t) \left( -(1 - e^{\frac{1}{2} - a\theta^{-1}})b + \frac{1}{2} + 2da\Lambda \right) \\ &\stackrel{(2.3.13)}{\leq} -\frac{1}{2}\psi(\rho, t). \end{aligned}$$

By using (2.3.15), we obtain the desired upper bound. Now assume that  $(2a)^{-1} \leq \rho \leq \theta^{-1}$ ,

$$\begin{aligned} \partial_t \phi(x, t) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(x, t)) &\leq \psi(\rho, t) \left( -(1 - e^{a(\rho - \theta^{-1})})b + a(1 - 4a\lambda)\rho + 2da\Lambda \right) \\ &\leq \psi(\rho, t) \left( -b + 2ad\Lambda + be^{a(\rho - \theta^{-1})} + a(1 - 4a\lambda)\rho \right). \end{aligned}$$

The function  $\rho \mapsto be^{a(\rho - \theta^{-1})} + a(1 - 4a\lambda)\rho$  is decreasing on  $] -\infty, \rho_0]$  and increasing on  $[\rho_0, +\infty[$  with  $\rho_0 := \frac{1}{a} \ln \left( \frac{4a\lambda - 1}{b} \right) + \theta^{-1}$ . By (2.3.13), the coefficients  $a$  and  $b$  are chosen so that  $\rho_0 < (2a)^{-1}$ . Under this assumption, the upper bound on the interval  $[(2a)^{-1}, \theta^{-1}]$  corresponds to  $\rho = \theta^{-1}$  which provides

$$\partial_t \phi(x, t) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(x, t)) \leq \psi(\rho, t)a(2d\Lambda + \theta^{-1} - 4\theta^{-1}a\lambda) \stackrel{(2.3.12)}{\leq} -\psi(\rho, t)a\theta^{-1} \stackrel{(2.3.12)}{\leq} -\psi(\rho, t).$$

By recalling (2.3.15), we obtain the desired upper bound.  $\square$

The following lemma contains the measure theoretic information necessary to conclude the proof of Proposition 2.3.1. The argument relies on Lemmas 2.3.3 and 2.3.4.

**Lemma 2.3.5.** *Let  $3/4 \leq \theta$ ,  $h_0 > 0$ ,  $\kappa_1 > 0$ ,  $A_\kappa$  be defined by (2.3.1) and  $G_{\theta, h_0}^-(x_0, t_0) \subseteq Q_1$ . There exist constants  $M \geq 1$  and  $\sigma > 0$  depending only on  $\theta$  and  $d, \lambda, \Lambda$  such that, if  $(x_0, t_0) \in A_{\kappa_1}$  and  $\kappa \geq \kappa_1$ , then*

$$|A_{M\kappa} \cap G_{\theta, h_0}^-(x_0, t_0)| \geq |G_{\theta, h_0}^-(x_0, t_0) \cap A_\kappa| + \sigma\eta |G_{\theta, h_0}^-(x_0, t_0) \setminus A_\kappa|.$$

*Proof.* We decompose the measure estimate into two parts

$$|A_{M\kappa} \cap G_{\theta, h_0}^-(x_0, t_0)| = |A_\kappa \cap G_{\theta, h_0}^-(x_0, t_0)| + |(A_{M\kappa} \setminus A_\kappa) \cap G_{\theta, h_0}^-(x_0, t_0)|.$$

It is enough to estimate  $|(A_{M\kappa} \setminus A_\kappa) \cap G_{\theta, h_0}^-(x_0, t_0)|$ . We claim that

$$|G_{\theta, h_0}^-(x_0, t_0) \setminus A_\kappa| \leq \frac{1}{\sigma\eta} |(A_{M\kappa} \setminus A_\kappa) \cap G_{\theta, h_0}^-(x_0, t_0)|.$$

For  $\kappa \geq \kappa_1$ , we define the collection of parabolic balls given by

$$\mathcal{B} := \left\{ G_{\theta,h}(x,t) : (x,t) \in G_{\theta,h_0}^-(x_0,t_0), \right. \\ \left. G_{\theta,h}(x,t) \cap G_{\theta,h_0}^-(x_0,t_0) \cap \{(y,s) : s < t+h\} \subseteq Q_1 \setminus A_\kappa \right. \\ \left. \text{and } G_{\theta,h}(x,t) \cap \{(y,s) : s = t+h\} \cap G_{\theta,h_0}^-(x_0,t_0) \cap A_\kappa \neq \emptyset \right\}.$$

Notice that for all  $(x,t) \in G_{\theta,h_0}^-(x_0,t_0)$ , the point  $(x_0,t_0)$  belongs to the parabolic ball  $G_{\theta,t_0-t}(x,t)$ . Observe that  $(x_0,t_0) \in A_\kappa$  by applying Lemma 2.3.2 with  $(x_0,t_0) \in A_{\kappa_1}$ . This implies that for all  $G_{\theta,h}(x,t) \in \mathcal{B}$ ,  $h \leq t_0 - t \leq h_0$ . Then, by Lemma 2.2.3, we may extract from  $\mathcal{B}$  a countable subcollection  $\{G_{\theta,h_i}(x_i,t_i) : i \in \mathbb{N}\}$  such that the  $G_{\theta,h_i}(x_i,t_i)$  are disjoint,

$$G_{\theta,h_0}^-(x_0,t_0) \setminus A_\kappa \subseteq \bigcup_{i \in \mathbb{N}} \widehat{G}_{\theta,h_i}(x_i,t_i) \quad \text{and} \quad \frac{|G_{\theta,h_i}(x_i,t_i)|}{|\widehat{G}_{\theta,h_i}(x_i,t_i)|} = \eta,$$

with  $\eta$  given by (2.2.3). By combining these, we get

$$|G_{\theta,h_0}^-(x_0,t_0) \setminus A_\kappa| \leq \sum_{i \in \mathbb{N}} |\widehat{G}_{\theta,h_i}(x_i,t_i)| = \frac{1}{\eta} \sum_{i \in \mathbb{N}} |G_{\theta,h_i}(x_i,t_i)|.$$

Next we complete the proof under the assumption that for all  $i \in \mathbb{N}$ ,

$$|G_{\theta,h_i}(x_i,t_i) \cap G_{\theta,h_0}^-(x_0,t_0) \cap A_{M\kappa}| \geq \sigma |G_{\theta,h_i}(x_i,t_i)| \quad (2.3.16)$$

for some constants  $M > 1$  and  $\sigma > 0$ , depending only on  $\theta, d, \lambda$  and  $\Lambda$ . Using also that the selected balls are disjoint, we obtain that

$$\sum_{i \in \mathbb{N}} |G_{\theta,h_i}(x_i,t_i)| \leq \frac{1}{\sigma} \sum_{i \in \mathbb{N}} |G_{\theta,h_i}(x_i,t_i) \cap G_{\theta,h_0}^-(x_0,t_0) \cap A_{M\kappa}| \\ = \frac{1}{\sigma} \left| \bigcup_{i \in \mathbb{N}} G_{\theta,h_i}(x_i,t_i) \cap \{(y,s) : s < t_i + h_i\} \cap G_{\theta,h_0}^-(x_0,t_0) \cap A_{M\kappa} \right|.$$

Since every ball in  $\mathcal{B}$  satisfies  $G_{\theta,h}(x,t) \cap \{(y,s) : s < t+h\} \cap G_{\theta,h_0}^-(x_0,t_0) \subseteq G_{\theta,h_0}^-(x_0,t_0) \setminus A_\kappa$ , we deduce that

$$|G_{\theta,h_0}^-(x_0,t_0) \setminus A_\kappa| \leq \frac{1}{\sigma\eta} |G_{\theta,h_0}^-(x_0,t_0) \cap (A_{M\kappa} \setminus A_\kappa)|.$$

The proof is complete, pending the verification of (2.3.16), which is achieved in the next lemma.  $\square$

The following lemma is the key step in the proof of Proposition 2.3.1.

**Lemma 2.3.6.** *Let  $3/4 \leq \theta$ ,  $\kappa_1 > 0$  and  $G_{\theta,h_0}^-(x_0,t_0) \subseteq Q_1$ . Suppose  $u \in \text{LSC}(Q_1)$  satisfies*

$$\partial_t u + \mathcal{P}_{\lambda,\Lambda}^+(D^2 u) \geq 0. \quad (2.3.17)$$

*There exist constants  $M \geq 1$  and  $\sigma > 0$  depending only on  $\theta$  and  $d, \lambda, \Lambda$  such that if  $(x_0,t_0) \in A_\kappa$ ,  $\kappa \geq \kappa_1$ , then for all  $(x,t) \in G_{\theta,h_0}^-(x_0,t_0)$  satisfying*

$$G_{\theta,h}(x,t) \cap \{(y,s) : s = t+h\} \cap G_{\theta,h_0}^-(x_0,t_0) \cap A_\kappa \neq \emptyset,$$

*we have*

$$|G_{\theta,h}(x,t) \cap A_{M\kappa} \cap G_{\theta,h_0}^-(x_0,t_0)| \geq \sigma |G_{\theta,h}(x,t)|.$$

*Proof.* Let  $(z_1, t_1) \in G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h\} \cap G_{\theta,h_0}^-(x_0, t_0) \cap A_\kappa$ . By the definition of  $A_\kappa$ , we can touch  $u$  at  $(z_1, t_1)$  from below by a concave paraboloid of opening  $\kappa$ : there exist  $y_1 \in B_1$  and  $-1 < s_1 \leq t_1$  such that

$$\begin{aligned} u(z_1, t_1) - \inf_{Q_1} u + \frac{\kappa}{2}|z_1 - y_1|^2 - \kappa(t_1 - s_1) \\ = \inf_{(z,\tau) \in Q_1, \tau \leq t_1} \left( u(z, \tau) - \inf_{Q_1} u + \frac{\kappa}{2}|z - y_1|^2 - \kappa(\tau - s_1) \right) = 0. \end{aligned} \quad (2.3.18)$$

**Step 1.** Let  $Q_r(x_2, t_2) \subseteq G_{\theta,h}(x, t) \cap G_{\theta,h_0}^-(x_0, t_0)$  be the cylinder given by Lemma 2.2.2 (see Figure 2.1). In particular,

$$t_2 = t + \frac{h}{2} \quad \text{and} \quad r \geq \frac{1}{\nu} \sqrt{\frac{h}{\theta}} \quad \text{with} \quad \nu = \frac{4}{\sqrt{2}-1}. \quad (2.3.19)$$

If we set

$$\alpha := \min \left\{ 1, \frac{\theta}{(1+\sqrt{2})^2} \right\} \quad \text{and} \quad \delta := \frac{1}{16\nu^2} \frac{\alpha}{\theta} > 0, \quad (2.3.20)$$

we claim that the parabolic ball (see Figure 2.1)

$$G_2 := G_{\alpha,h(\frac{1}{2}+\delta)}(x_2, t_2 - \delta h)$$

satisfies

$$G_2 \cap \{(y, s) : s \leq t_2\} \subseteq \overline{Q}_{r/4}(x_2, t_2) \quad (2.3.21)$$

and

$$(z_1, t_1) \in (G_2 \setminus \partial_p G_2) \cap \{(y, s) : s = t_1\}. \quad (2.3.22)$$

To obtain the first assertion, it suffices to show that the vertex  $(x_2, t_2 - \delta h)$  of  $G_2$  is in  $\overline{Q}_{r/4}(x_2, t_2)$  and  $G_2 \cap \{(y, s) : s = t_2\} \subseteq \overline{B}_{r/4}(x_2)$ . First, by using (2.3.19) and (2.3.20), the inequality

$$\delta h \leq \left( \frac{1}{16\nu^2} \frac{\alpha}{\theta} \right) \theta \nu^2 r^2 = \frac{\alpha}{16} r^2 \leq \frac{r^2}{16}$$

implies  $(x_2, t_2 - \delta h) \in \overline{Q}_{r/4}(x_2, t_2)$ . By the definition of the parabolic ball  $G_2$  and using (2.3.19) and (2.3.20), each  $(z, t_2) \in G_2$  satisfies  $|z - x_2|^2 \leq \alpha^{-1} \delta h \leq \frac{r^2}{16}$ . Then, for the second assertion, observing that  $(z_1, t_1) \in G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h\}$ , it is enough to show that

$$G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h\} \subseteq (G_2 \setminus \partial_p G_2) \cap \left\{ (y, s) : s = t_2 + \frac{h}{2} \right\}.$$

Let  $(z, \tau) \in G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h\}$ . Since  $(x_2, t_2) \in G_{\theta,h}(x, t) \cap \{(y, s) : s = t + h/2\}$  and  $\delta > 0$ , we get

$$|z - x_2| \leq |z - x| + |x - x_2| = \left( 1 + \frac{1}{\sqrt{2}} \right) \sqrt{\frac{h}{\theta}} \stackrel{(2.3.20)}{\leq} \sqrt{\left( \frac{1}{2} + \delta \right) \frac{h}{\alpha}}.$$

This is equivalent to  $(z, \tau) \in (G_2 \setminus \partial_p G_2) \cap \{(y, s) : s = t_2 + \frac{h}{2}\}$ , which gives (2.3.22).

**Step 2.** We claim that there exists  $(z_2, t_2) \in G_2 \cap \{(y, s) : s = t_2\}$  such that

$$u(z_2, t_2) - \inf_{Q_1} u + \frac{\kappa}{2}|z_2 - y_1|^2 - \kappa(t_2 - s_1) \leq (d\Lambda + 3)\kappa\beta\theta\nu^2 r^2. \quad (2.3.23)$$

By applying (a properly scaled) Lemma 2.3.4, there exists a barrier function  $w$  which satisfies

$$\begin{cases} \partial_t w + \mathcal{P}_{\lambda,\Lambda}^+(D^2 w) \leq -1, & \text{in } G_2 \cap \{(y, s) : s > t_2\}, \\ w = 0, & \text{on } \partial_p G_2 \cap \{(y, s) : s > t_2\}, \\ 0 \leq w \leq \beta h, & \text{on } G_2 \cap \{(y, s) : s = t_2\}, \end{cases} \quad (2.3.24)$$



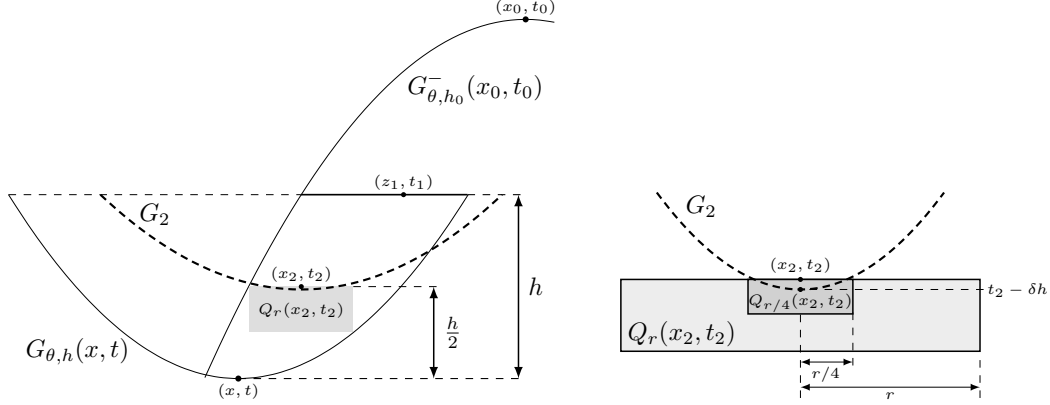


Figure 2.1:  $Q_r(x_2, t_2) \subseteq G_{\theta,h}(x, t) \cap G_{\theta,h_0}^-(x_0, t_0)$  and  $G_2 = G_{\alpha,h(1/2+\delta)}(x_2, t_2 - \delta h)$ .

and  $w > 0$  in  $(G_2 \setminus \partial_p G_2) \cap \{(y, s) : s > t_2\}$ . In particular, this implies by Step 1 that  $w(z_1, t_1) > 0$ . We have that  $w \leq \beta h$  in  $G_2 \cap \{(y, s) : s > t_2\}$  by the maximum principle. Observe that the function

$$\varphi(z, \tau) := ((d\Lambda + 1)\kappa + 2\kappa_1)w - \frac{\kappa}{2}|z - y_1|^2 + \kappa\tau,$$

satisfies

$$\partial_t \varphi + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \varphi) \leq -2\kappa_1.$$

Notice that  $u$  satisfies

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -\kappa_1. \quad (2.3.25)$$

The comparison principle implies that the map  $(z, \tau) \mapsto u(z, \tau) - \varphi(z, \tau)$  attains its infimum in  $G_2 \cap \{(y, s) : s > t_2\}$  at some point  $(z, \tau) = (z_2, t_2) \in \partial_p(G_2 \cap \{(y, s) : s > t_2\})$ . It is impossible that  $(z_2, t_2) \in \partial_p G_2 \cap \{(y, s) : s > t_2\}$  since (2.3.18),  $w = 0$  on  $\partial_p G_2 \cap \{(y, s) : s > t_2\}$  and  $w(z_1, t_1) > 0$  imply that

$$\begin{aligned} u(z_1, t_1) - \varphi(z_1, t_1) &= u(z_1, t_1) + \frac{\kappa}{2}|z_1 - y_1|^2 - \kappa t_1 - ((d\Lambda + 1)\kappa + 2\kappa_1)w(z_1, t_1) \\ &\stackrel{(2.3.18)}{<} \inf_{\substack{(z, \tau) \in Q_1 \\ \tau \leq \tau_1}} (u(z, \tau) + \frac{\kappa}{2}|z - y_1|^2 - \kappa\tau) \\ &\leq \inf_{\substack{(z, \tau) \in \partial_p G_2 \\ t_2 < \tau \leq t_1}} (u(z, \tau) + \frac{\kappa}{2}|z - y_1|^2 - \kappa\tau) = \inf_{\substack{(z, \tau) \in \partial_p G_2 \\ t_2 < \tau \leq t_1}} (u(z, \tau) - \varphi(z, \tau)). \end{aligned}$$

Moreover, it is impossible that  $(z_2, t_2)$  satisfies  $t_2 = t_1$  since  $\varphi$  satisfies

$$\partial_t \varphi(\cdot, t_1) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \varphi(\cdot, t_1)) \leq -2\kappa_1 \quad \text{for } G_2 \cap \{(y, s) : s = t_1\}.$$

Hence  $(z_2, t_2) \in G_2 \cap \{(y, s) : s = t_2\}$  and so, in particular, by (2.3.21) satisfies  $|z_2 - x_2| \leq \frac{1}{4}r$ , and

$$\varphi(z_2, t_2) = ((d\Lambda + 1)\kappa + 2\kappa_1)w(z_2, t_2) - \frac{\kappa}{2}|z_2 - y_1|^2 + \kappa t_2.$$

Using that  $w > 0$  in  $G_2 \cap \{(y, s) : s > t_2\}$ , we obtain that

$$\begin{aligned} u(z_1, t_1) + \frac{\kappa}{2}|z_1 - y_1|^2 - \kappa t_1 &\geq u(z_1, t_1) - \varphi(z_1, t_1) \\ &\geq \inf_{G_2 \cap \{s > t_2\}} (u(z, \tau) - \varphi(z, \tau)) = u(z_2, t_2) - \varphi(z_2, t_2) \\ &= u(z_2, t_2) + \frac{\kappa}{2}|z_2 - y_1|^2 - \kappa t_2 - ((d\Lambda + 1)\kappa + 2\kappa_1)w(z_2, t_2). \end{aligned}$$

By combining (2.3.19) and (2.3.24), we know that

$$w(z_2, t_2) \leq \beta \theta \nu^2 r^2.$$

Using this together with (2.3.18),  $(z_1, t_1) \in (G_2 \setminus \partial_p G_2) \cap \{(y, s) : s = t_2 + \frac{h}{2}\}$  (by (2.3.22)) and  $\kappa \geq \kappa_1$ ,

$$\inf_{(z, \tau) \in Q_1, \tau \leq t_1} \left( u(z, \tau) + \frac{\kappa}{2} |z - y_1|^2 - \kappa \tau \right) \geq u(z_2, t_2) + \frac{\kappa}{2} |z_2 - y_1|^2 - \kappa t_2 - (d\Lambda + 3) \kappa \beta \theta \nu^2 r^2.$$

Recalling (2.3.18), we obtain (2.3.23).

**Step 3.** Let  $\gamma := 17(d\Lambda + 3)\beta\theta\nu^2$ . We claim that for all  $(y_2, s_2) \in G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16})$ , there exists a point  $(z_3, t_3) \in G_{1/2, t_2-s_2}(y_2, s_2) \subseteq Q_r(x_2, t_2)$  such that

$$\chi(z_3, t_3) = \inf_{\substack{(z, \tau) \in Q_1 \\ \tau \leq t_3}} \chi(z, \tau) = 0, \quad (2.3.26)$$

where the function  $\chi$  is given by

$$\chi(z, \tau) := u(z, \tau) - \inf_{Q_1} u + \frac{\kappa}{2} |z - y_1|^2 - \kappa(\tau - s_1) + \frac{\gamma\kappa}{2} |z - y_2|^2 - \gamma\kappa(\tau - s_2).$$

Observe the function  $\chi$  can be written in the form

$$\chi(z, \tau) = u(z, \tau) - \inf_{Q_1} u - P_{y_1, s_1; \kappa}(z, \tau) - P_{y_2, s_2; \gamma\kappa}(z, \tau)$$

where  $P_{y_1, s_1; \kappa}$  and  $P_{y_2, s_2; \gamma\kappa}$  are the concave paraboloids

$$\begin{cases} P_{y_1, s_1; \kappa}(z, \tau) := -\frac{\kappa}{2} |z - y_1|^2 + \kappa(\tau - s_1) \\ P_{y_2, s_2; \gamma\kappa}(z, \tau) := -\frac{\gamma\kappa}{2} |z - y_2|^2 + \gamma\kappa(\tau - s_2). \end{cases}$$

Let  $(y_2, s_2) \in G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16})$ . By combining Steps 1-2, observe that

$$(z_2, t_2) \in G_2 \cap \{(y, s) : s = t_2\} \subseteq \overline{Q}_{r/4}(x_2, t_2).$$

By applying Lemma 2.2.2 part (P3), we get  $G_{1/2, t_2-s_2}(y_2, s_2) \subseteq Q_r(x_2, t_2)$ . Define

$$\zeta(t) := \inf \{ \chi(z, \tau) : (z, \tau) \in G_{1/2, t_2-s_2}(y_2, s_2), \tau \leq t \}.$$

To prove the claim, we will use the three following facts:

$$t \mapsto \inf \{ \chi(z, \tau) : (z, \tau) \in Q_1 \setminus G_{1/2, t_2-s_2}(y_2, s_2), \tau \leq t \} \text{ is nonnegative on } [s_2, t_2], \quad (2.3.27)$$

and

$$\chi(z_2, t_2) < 0 \leq \chi(y_2, s_2), \quad (2.3.28)$$

and

$$\begin{aligned} t \mapsto \zeta(t) \text{ is non-increasing and right-continuous on } [s_2, t_2], \\ \text{and left-continuous on } [s_2, t_2] \cap \{s : \zeta(s) < 0\}. \end{aligned} \quad (2.3.29)$$

Assuming that we have shown (2.3.27), (2.3.28) and (2.3.29), let us prove the claim given by (2.3.26). Then for all  $t \in [s_2, t_2]$ ,

$$\inf \{ \chi(z, \tau) : (z, \tau) \in Q_1, \tau \leq t \} = \min \left( \zeta(t), \inf \{ \chi(z, \tau) : (z, \tau) \in Q_1 \setminus G_{1/2, t_2-s_2}(y_2, s_2), \tau \leq t \} \right).$$

By (2.3.27), the second infimum in the right-hand side above is nonnegative for  $t \in [s_2, t_2]$  and it suffices to study the sign of  $\zeta(t)$ . First notice  $\zeta(s_2) = \chi(y_2, s_2) \geq 0$  by (2.3.28). Then, since  $(z_2, t_2) \in G_{1/2, t_2-s_2}(y_2, s_2)$  we obtain also by (2.3.28) that

$$\zeta(t_2) \leq \chi(z_2, t_2) < 0.$$

By (2.3.29), we deduce there exists a time  $t_3 \in [s_2, t_2]$  such that

$$\zeta(t_3) = \inf \{ \chi(z, \tau) : (z, \tau) \in G_{1/2, t_2 - s_2}(y_2, s_2) \text{ and } \tau \leq t_3 \} = 0.$$

Since  $\chi \in \text{LSC}(Q_1)$ , there exists  $(z_3, t_3) \in G_{1/2, t_2 - s_2}(y_2, s_2)$  realizing this infimum, and so, satisfying (2.3.26). Then, the inclusion  $G_{1/2, t_2 - s_2}(y_2, s_2) \subseteq Q_r(x_2, t_2)$  yields the claim. To complete the proof of (2.3.26), it remains to check (2.3.27), (2.3.28) and (2.3.29). To get (2.3.27), observe that, for all  $(z, \tau) \in Q_1 \setminus G_{1/2, t_2 - s_2}(y_2, s_2)$ ,  $\tau \leq t_2$ ,

$$\tau - s_2 - \frac{|z - y_2|^2}{2} < 0,$$

and so,

$$\chi(z, \tau) > u(z, \tau) - \inf_{Q_1} u - P_{y_1, s_1; \kappa}(z, \tau).$$

By (2.3.18), the right-hand side in the inequality above is nonnegative. Therefore, we have  $\chi(z, \tau) > 0$  for all  $(z, \tau) \in Q_1 \setminus G_{1/2, t_2 - s_2}(y_2, s_2)$ ,  $\tau \leq t_2 \leq t_1$ . By passing to the inf, we obtain (2.3.27). By repeating (2.3.18), we also obtain that

$$\chi(y_2, s_2) = u(y_2, s_2) - \inf_{Q_1} u - P_{y_1, s_1; \kappa}(y_2, s_2) \geq 0.$$

For (2.3.28), since  $(y_2, s_2) \in G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16})$ , we have

$$t_2 - s_2 - \frac{|y_2 - z_2|^2}{2} \geq \frac{r^2}{16}.$$

This implies  $P_{y_2, s_2; \gamma \kappa}(z_2, t_2) \geq \gamma \kappa \frac{r^2}{16}$ . By using also Step 2, we obtain

$$\chi(z_2, t_2) < (d\Lambda + 3)\kappa\beta\theta\nu^2 r^2 - \gamma \kappa \frac{r^2}{16}.$$

By inserting the value of  $\gamma$ , we get

$$(d\Lambda + 3)\kappa\beta\theta\nu^2 r^2 - \gamma \kappa \frac{r^2}{16} = \frac{\kappa r^2}{16} (16(d\Lambda + 3)\beta\theta\nu^2 - \gamma) < 0,$$

and the claim (2.3.28) follows.

For (2.3.29), it is clear that  $\zeta$  is non-increasing. Moreover,  $\zeta \in \text{LSC}(Q_1)$  since  $u \in \text{LSC}(Q_1)$ . The lower semicontinuity and the monotonicity of  $\zeta$  imply that  $\zeta$  is right-continuous. To show that  $\zeta$  is left-continuous, we argue by contradiction. Assume there exist  $\bar{t} \in (s_2, t_2]$  and a strictly increasing sequence  $r_k \rightarrow \bar{t}$  such that

$$\zeta(r_k) \xrightarrow[k \rightarrow +\infty]{} \zeta^-(\bar{t}) > \zeta(\bar{t}), \quad \text{with } \zeta(\bar{t}) < 0, \quad (2.3.30)$$

where  $\zeta^-(\tau)$  denotes the limit from the left of  $\zeta$  at  $\tau$ . Define

$$P(z, \tau) := P_{y_1, s_1; \kappa}(z, \tau) + P_{y_2, s_2; \gamma \kappa}(z, \tau).$$

By (2.3.30), we deduce that for all  $(z, \tau) \in G_{1/2, t_2 - s_2}(y_2, s_2)$ ,  $\tau < \bar{t}$ ,

$$u(z, \tau) - \inf_{Q_1} u - P(z, \tau) \geq \zeta^-(\bar{t}), \quad (2.3.31)$$

and there exists  $(\bar{z}, \bar{t}) \in G_{1/2, t_2 - s_2}(y_2, s_2)$  such that

$$\zeta(\bar{t}) = u(\bar{z}, \bar{t}) - \inf_{Q_1} u - P(\bar{z}, \bar{t}) = \chi(\bar{z}, \bar{t}) = \inf \{ \chi(z, \tau) : (z, \tau) \in G_{1/2, t_2 - s_2}(y_2, s_2), \tau \leq \bar{t} \}. \quad (2.3.32)$$

Notice that by using (2.3.18),  $\zeta(\bar{t}) < 0$  implies that  $(\bar{z}, \bar{t}) \in G_{1/2, t_2 - s_2}(y_2, s_2) \setminus \partial_p G_{1/2, t_2 - s_2}(y_2, s_2)$ . Let  $\phi$  be a smooth function such that  $u - \phi$  has a local minimum at  $(\bar{z}, \bar{t})$ . Denote  $\tilde{\phi}$  the map  $\tilde{\phi}(y, s) := \phi(y, s) - L(s - \bar{t})$  with  $L \geq 0$  to be selected below. For  $s \geq \bar{t}$ , using that  $\tilde{\phi}(\bar{z}, \bar{t}) = \phi(\bar{z}, \bar{t})$ , we obtain

$$u(y, s) - \tilde{\phi}(y, s) = (u - \phi)(y, s) + L(s - \bar{t}) \geq (u - \phi)(\bar{z}, \bar{t}). \quad (2.3.33)$$

Consider now  $s < \bar{t}$ ,

$$\begin{aligned} u(y, s) - \tilde{\phi}(y, s) &\stackrel{(2.3.31)}{\geq} \zeta^-(\bar{t}) + \inf_{Q_1} u + P(y, s) - \phi(y, s) + L(s - \bar{t}) \\ &\stackrel{(2.3.32)}{\geq} u(\bar{z}, \bar{t}) - \tilde{\phi}(\bar{z}, \bar{t}) + P(y, s) - P(\bar{z}, \bar{t}) - (\phi(y, s) - \phi(\bar{z}, \bar{t})) + (\zeta^-(\bar{t}) - \zeta(\bar{t})) + L(s - \bar{t}). \end{aligned}$$

On the set

$$\left\{ (y, s) \in G_{1/2, t_2-s_2}(y_2, s_2) \setminus \partial_p G_{1/2, t_2-s_2}(y_2, s_2) : s \leq \bar{t} \text{ and } \max \{ |P(y, s) - P(\bar{z}, \bar{t})|, |\phi(y, s) - \phi(\bar{z}, \bar{t})|, L|s - \bar{t}| \} \leq \frac{1}{4}(\zeta^-(\bar{t}) - \zeta(\bar{t})) \right\},$$

the following inequality holds true

$$u(y, s) - \tilde{\phi}(y, s) > u(\bar{z}, \bar{t}) - \tilde{\phi}(\bar{z}, \bar{t}) + \frac{1}{4}(\zeta^-(\bar{t}) - \zeta(\bar{t})), \quad (2.3.34)$$

and by putting together the two cases (2.3.33) and (2.3.34), the map  $(y, s) \mapsto (u - \tilde{\phi})(y, s)$  has a local minimum at  $(\bar{z}, \bar{t})$ . Using that  $u$  is a supersolution of (2.3.17), we obtain

$$0 \leq \partial_t \tilde{\phi}(\bar{z}, \bar{t}) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \tilde{\phi}(\bar{z}, \bar{t})) = -L + \partial_t \phi(\bar{z}, \bar{t}) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(\bar{z}, \bar{t})).$$

Taking  $L := 1 + \partial_t \phi(\bar{z}, \bar{t}) + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi(\bar{z}, \bar{t})) \geq 1$ , we get a contradiction.

**Step 4.** Consider the function  $\bar{Z} : G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16}) \rightarrow Q_1$  given by  $\bar{Z}(y, s) = (\bar{z}(y, s), \bar{\tau}(y, s))$ , where

$$\begin{cases} \bar{z}(y, s) := \frac{1}{\gamma+1}(y_1 + \gamma y), \\ \bar{\tau}(y, s) := \frac{1}{\gamma+1} \left( s_1 + \gamma s - \frac{\gamma}{(\gamma+1)^2} |y_1 - y|^2 \right). \end{cases}$$

Observe by completing the square that we have for all  $z \in \mathbb{R}^d, \tau \in \mathbb{R}$ ,

$$\frac{\kappa}{2} |z - y_1|^2 - \kappa(\tau - s_1) + \frac{\gamma\kappa}{2} |z - y_2|^2 - \gamma\kappa(\tau - s_2) = \frac{(\gamma+1)\kappa}{2} |z - \bar{z}(y_2, s_2)|^2 - (\gamma+1)\kappa(\tau - \bar{\tau}(y_2, s_2)).$$

It follows by Step 3 that the map

$$(z, \tau) \mapsto u(z, \tau) - \inf_{Q_1} u + \frac{(\gamma+1)}{2} |z - \bar{z}(y_2, s_2)|^2 - (\gamma+1)(\tau - \bar{\tau}(y_2, s_2))$$

attains its infimum in  $Q_1$  at some point of  $Q_r(x_2, t_2)$  and this infimum is equal to zero. Since  $u$  satisfies (2.3.25), we can apply Lemma 2.3.3 by taking  $L = \kappa_1 > 0$  and we obtain

$$\left| \bar{Z} \left( G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16}) \right) \right| \leq \frac{1}{\lambda^d} \left( 1 + \frac{\kappa_1}{(\gamma+1)\kappa} + \Lambda d \right)^{d+1} |Q_r(x_2, t_2) \cap A_{(\gamma+1)\kappa}|. \quad (2.3.35)$$

Since  $\gamma \geq 1$  and  $\gamma/(\gamma+1) \geq \frac{1}{2}$ , we deduce by the change of variables formula that

$$\left| \bar{Z} \left( G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16}) \right) \right| \geq 2^{-d-1} \left| G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16}) \right|. \quad (2.3.36)$$

By combining the explicit expression of a parabolic ball and (2.2.2), we deduce there exists a numerical constant  $0 < c < 1$  such that

$$\left| G_{1/2, r^2/16}^-(z_2, t_2 - \frac{r^2}{16}) \right| \geq \frac{c}{\theta} |G_{\theta, h}(x, t)|. \quad (2.3.37)$$

By Lemma 2.2.2 part (P1),  $Q_r(x_2, t_2) \subseteq G_{\theta, h}(x, t) \cap G_{\theta, h_0}^-(x_0, t_0)$ . By combining this observation with (2.3.35), (2.3.36) and (2.3.37), recalling that  $\kappa \geq \kappa_1$  and taking  $M := \gamma + 1$ , we obtain that

$$|G_{\theta, h}(x, t) \cap A_{M\kappa} \cap G_{\theta, h_0}^-(x_0, t_0)| \geq \underbrace{\frac{2^{-d-1}\lambda^d}{(1 + \frac{1}{\gamma+1} + \Lambda d)^{d+1}}}_{:=\sigma} \frac{c}{\theta} |G_{\theta, h}(x, t)|,$$

as desired.  $\square$

We next present the proof of the parabolic  $W^{2, \varepsilon}$  estimate.

*Proof of Proposition 2.3.1.* We begin with four reductions. First, we may assume that  $L = 0$ . Otherwise we replace  $u$  by  $\hat{u} := u + Lt$ , which is solution of

$$\partial_t \hat{u} + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \hat{u}) \geq 0.$$

Let  $(x, t) \in Q_1$  and  $A > \underline{\Theta}(\hat{u}, Q_1)(x, t)$ . By the definition of  $\underline{\Theta}(\hat{u}, Q_1)(x, t)$ ,

$$u(y, s) + Ls \geq u(x, t) + Lt + p \cdot (x - y) - A \left( \frac{1}{2} |x - y|^2 + (t - s) \right).$$

Since  $L \geq 0$  and  $t - s \geq 0$ , we deduce that, for all  $(y, s) \in Q_1$ ,  $s \leq t$ ,

$$u(y, s) \geq u(x, t) + p \cdot (x - y) - A \left( \frac{1}{2} |x - y|^2 + (t - s) \right).$$

Then  $A \geq \underline{\Theta}(u, Q_1)(x, t)$ , and so  $\underline{\Theta}(\hat{u}, Q_1)(x, t) \geq \underline{\Theta}(u, Q_1)(x, t)$ . Under the assumption the estimate holds true for  $\hat{u}$ , we get

$$\begin{aligned} |\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u, Q_1)(x, t) > \kappa\}| &\leq |\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(\hat{u}, Q_1)(x, t) > \kappa\}| \\ &\leq C \left( \frac{\kappa}{\sup_{Q_1} |\hat{u}|} \right)^{-\varepsilon}. \end{aligned}$$

Observing that  $\sup_{Q_1} |\hat{u}| \leq \sup_{Q_1} |u| + L$ , we get the desired estimate for  $u$ .

Next, using the positive homogeneity of  $\mathcal{P}_{\lambda, \Lambda}^+$  and  $\underline{\Theta}$  and replacing  $u$  by  $\tilde{u} := u / \sup_{Q_1} |u|$ , we may assume that  $\sup_{Q_1} |u| = 1$ . Finally, by Lemma 2.3.2, we have

$$|\{(x, s) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u, Q_1)(x, s) > \kappa\}| \leq |Q_{1/2}(0, -\tfrac{1}{4}) \setminus A_\kappa|.$$

Thus, it suffices to prove that there exist some universal constants  $C$ ,  $\kappa_0$ ,  $\varepsilon > 0$  such that for all  $\kappa \geq \kappa_0$ ,

$$|Q_{1/2}(0, -\tfrac{1}{4}) \setminus A_\kappa| \leq C \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon}. \quad (2.3.38)$$

**Step 1** (Geometric configuration). Fix  $\frac{3}{4} \leq \theta \leq 5$ . We consider the cube

$$C_R(0, T_0) := (0 + (-R, R)^d) \times (T_0 - H_{\text{cube}}, T_0],$$

with  $T_0 \in (-\frac{1}{2}, -\frac{1}{4}]$ ,  $R > 0$  which satisfies

$$R < \frac{1}{(2 + 16\theta)\sqrt{d}}, \quad (2.3.39)$$

and  $H_{\text{cube}}$ , depending on  $R$  and  $\theta$ , given by

$$H_{\text{cube}} := (1 + 5\theta)dR^2.$$

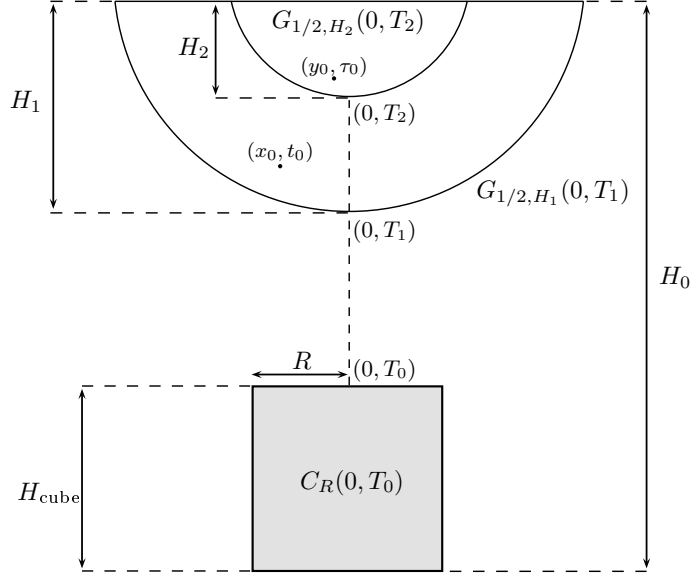


Figure 2.2: Geometry in the proof of Proposition 2.3.1: the cube  $C_R(0, T_0)$  and the parabolic balls  $G_{1/2, H_1}(0, T_1)$  and  $G_{1/2, H_2}(0, T_2)$  such that  $C_R(0, T_0) \subseteq G_{\theta, H_0}^-(x_0, t_0)$  with  $(x_0, t_0) \in G_{1/2, H_1}(0, T_1)$ .

To perform our analysis, we introduce two parabolic balls  $G_{1/2, H_1}(0, T_1)$  and  $G_{1/2, H_2}(0, T_2)$  (see Figure 2.2) where the parameters  $T_1$ ,  $H_1$ ,  $T_2$  and  $H_2$  denote

$$\begin{aligned} H_1 &:= \frac{1}{2}dR^2 \quad \text{and} \quad T_1 := T_0 + 4d\theta R^2, \\ H_2 &:= \frac{1}{8}dR^2 \quad \text{and} \quad T_2 := T_1 + H_1 - H_2 = T_1 + \frac{3}{8}dR^2, \end{aligned}$$

and a parabolic ball  $G_{\theta, H_0}^-(x_0, t_0)$  with  $(x_0, t_0)$  selected at Step 2 and

$$H_0 := T_1 - T_0 + H_1 + H_{\text{cube}} = \left(\frac{3}{2} + 9\theta\right)dR^2.$$

To be well-defined, our argument requires that  $G_{1/2, H_1}(0, T_1)$  and  $G_{1/2, H_2}(0, T_2)$  are contained in  $Q_1$ . In particular, we need to check the condition  $T_1 + H_1 < 0$ . By using the explicit expressions of  $T_1$ ,  $H_1$  and (2.3.39), we get

$$T_1 + H_1 = T_0 + 4d\theta R^2 + \frac{1}{2}dR^2 \leq -\frac{1}{4} + \left(\frac{1}{2} + 4\theta\right)dR^2 < 0.$$

**Step 2** (Existence of the paraboloid for a certain  $\kappa = \kappa_0$ ). We claim that there exists  $(x_0, t_0) \in G_{1/2, H_1}(0, T_1)$  such that there exists  $(y_0, s_0) \in B_1 \times (-1, t_0]$  such that

$$\begin{aligned} u(x_0, t_0) - \inf_{Q_1} u + \frac{\kappa}{2}|x_0 - y_0|^2 - \kappa(t_0 - s_0) \\ = \inf_{(z, \tau) \in Q_1, \tau \leq t_0} \left( u(z, \tau) - \inf_{Q_1} u + \frac{\kappa}{2}|z - y_0|^2 - \kappa(\tau - s_0) \right) = 0. \end{aligned} \quad (2.3.40)$$

In addition,

$$C_R(0, T_0) \subseteq G_{\theta, H_0}^-(x_0, t_0) \subseteq Q_1, \quad (2.3.41)$$

and there exists  $0 < \xi < 1$ , depending only on  $\theta$ , such that

$$|C_R(0, T_0)| \geq \xi |G_{\theta, H_0}^-(x_0, t_0)|. \quad (2.3.42)$$

To prove the claim given by (2.3.40), first we are going to find  $(x_0, t_0)$  realizing the infimum for a good choice of  $y_0$  and  $\kappa$  chosen sufficiently large. Then we are going to determine  $s_0$  such that this infimum will be equal to zero and we are going to check that  $-1 < s_0 \leq t_0$  to complete the proof.

First select  $(y_0, \tau_0) \in G_{1/2, H_2}(0, T_2)$  such that

$$u(y_0, \tau_0) = \inf \{u(z, \tau) : (z, \tau) \in G_{1/2, H_2}(0, T_2)\},$$

and  $\kappa$  such that

$$\kappa \geq \kappa_0 \quad \text{with} \quad \kappa_0 := \max \left\{ 24, \frac{320}{dR^2} \right\}. \quad (2.3.43)$$

Then for all  $(z, \tau) \in Q_1$ ,

$$u(z, \tau) + \frac{\kappa}{2}|z - y_0|^2 - \kappa\tau \geq u(y_0, \tau_0) - \kappa\tau_0 - \text{osc}_{Q_1} u + \kappa \left( \frac{1}{2}|z - y_0|^2 - (\tau - \tau_0) \right).$$

We check that  $T_2 - T_1 > 2H_2$ . For all  $(z, \tau) \in Q_1 \setminus G_{1/2, H_1}(0, T_1)$ ,  $\tau \leq \tau_0$ , we have

$$\frac{1}{2}|z - y_0|^2 - (\tau - \tau_0) \geq \begin{cases} \frac{1}{2}(\sqrt{T_2 - T_1} - \sqrt{2H_2})^2, & \text{if } \tau \in [\frac{1}{2}(T_1 + T_2), \tau_0], \\ -\frac{1}{2}(T_1 + T_2) + \tau_0, & \text{if } \tau < \frac{1}{2}(T_1 + T_2). \end{cases}$$

By inserting the values of  $T_1, T_2$  and by using that  $\tau_0 \in G_{1/2, H_2}(0, T_2)$ , we get

$$\frac{1}{2}|z - y_0|^2 - (\tau - \tau_0) \geq \begin{cases} \frac{1}{160}dR^2, & \text{if } \tau \in [\frac{1}{2}(T_1 + T_2), \tau_0], \\ \frac{3}{16}dR^2, & \text{if } \tau < \frac{1}{2}(T_1 + T_2). \end{cases}$$

Since  $\text{osc}_{Q_1} u \leq 2$ , and recalling (2.3.43), this implies that

$$\inf \{u(z, \tau) + \frac{\kappa}{2}|z - y_0|^2 - \kappa\tau : (z, \tau) \in Q_1 \setminus G_{1/2, H_1}(0, T_1), \tau \leq \tau_0\} \geq u(y_0, \tau_0) - \kappa\tau_0.$$

Thus there exists  $(x_0, t_0) \in G_{1/2, H_1}(0, T_1)$ ,  $t_0 \leq \tau_0$ , such that

$$\begin{aligned} u(x_0, t_0) + \frac{\kappa}{2}|x_0 - y_0|^2 - \kappa t_0 &= \inf_{(z, \tau) \in Q_1, \tau \leq \tau_0} (u(z, \tau) + \frac{\kappa}{2}|z - y_0|^2 - \kappa\tau) \\ &= \inf_{(z, \tau) \in Q_1, \tau \leq t_0} (u(z, \tau) + \frac{\kappa}{2}|z - y_0|^2 - \kappa\tau). \end{aligned}$$

Now let  $s_0 \leq t_0$  be defined by

$$s_0 := t_0 - \frac{1}{\kappa} \left( u(x_0, t_0) - \inf_{Q_1} u \right) - \frac{1}{2}|x_0 - y_0|^2.$$

Thus, the infimum in (2.3.40) is equal to zero. To complete the proof of the claim, it remains to check that  $s_0 > -1$ . By using that  $u(x_0, t_0) - \inf_{Q_1} u \leq 2$ , we get

$$t_0 - s_0 \leq \frac{2}{\kappa} + \frac{1}{2}|x_0 - y_0|^2.$$

Since  $(x_0, t_0), (y_0, \tau_0) \in G_{\theta, H_1}(0, T_1)$ , we get by inserting (2.3.39) and using  $\theta \geq 3/4$  that

$$|x_0 - y_0|^2 \leq \frac{2H_1}{\theta} = \frac{dR^2}{\theta} \leq \frac{1}{4\theta(1+8\theta)^2} \leq \frac{1}{147} \leq \frac{1}{6}.$$

Thus we obtain

$$t_0 - s_0 \leq \frac{2}{\kappa} + \frac{1}{12}.$$

By using (2.3.43), we get

$$t_0 - \frac{1}{6} \leq s_0 \leq t_0.$$

Since  $t_0 \in [T_1, T_1 + H_1] = [T_0 + 4d\theta R^2, T_0 + (4\theta + \frac{1}{2})dR^2]$ , in particular,  $-\frac{1}{2} \leq T_0 \leq t_0 \leq 0$  and we conclude that  $s_0 > -1$ . This completes the proof of (2.3.40).

To show (2.3.41), first notice that  $|x_0| \leq \sqrt{d}R$  since  $(x_0, t_0) \in G_{1/2, H_1}(0, T_1)$ . Now observe that each  $(y, s) \in C_R(0, T_0)$  satisfies  $t_0 - s \leq H_0$  and

$$|y - x_0| \leq |y| + |x_0| \leq \sqrt{d}R + \sqrt{d}R = 2\sqrt{d}R.$$

This means  $(y, s) \in G_{\theta, H_0}^-(x_0, t_0)$  since  $T_1 - T_0 \geq 4d\theta R^2$  (see also Figure 2.2). By a direct computation, we check (2.3.42).

**Step 3.** By Step 2, the point  $(x_0, t_0)$  belongs to  $A_{\kappa_0}$ . By Lemma 2.3.2, we deduce that, for all  $\kappa \geq \kappa_0$ ,  $(x_0, t_0)$  belongs to  $A_\kappa$ . Then we can apply Lemma 2.3.5 and we get that for all  $\kappa \geq \kappa_0$ ,

$$|A_{M\kappa} \cap G_{\theta, H_0}^-(x_0, t_0)| \geq |G_{\theta, H_0}^-(x_0, t_0) \cap A_\kappa| + \sigma\eta |G_{\theta, H_0}^-(x_0, t_0) \setminus A_\kappa|.$$

After rearranging the terms, this implies

$$|G_{\theta, H_0}^-(x_0, t_0) \setminus A_{M\kappa}| \leq (1 - \sigma\eta) |G_{\theta, H_0}^-(x_0, t_0) \setminus A_\kappa|. \quad (2.3.44)$$

**Step 4.** We claim that for all  $\kappa > \kappa_0$ , we have

$$|C_R(0, T_0) \setminus A_\kappa| \leq \frac{1}{\xi(1 - \sigma\eta)} \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon} |C_R(0, T_0)|,$$

where  $\varepsilon$  is given by  $\varepsilon := -\frac{\ln(1 - \sigma\eta)}{\ln M} > 0$ . First we obtain the decay measure estimate on the parabolic ball  $G_{\theta, H_0}^-(x_0, t_0)$ . Let  $\kappa > \kappa_0$  and  $N$  be the integer defined by

$$M^N \kappa_0 < \kappa \leq M^{N+1} \kappa_0 \iff N := \left\lceil \frac{1}{\ln M} \ln \left( \frac{\kappa}{\kappa_0} \right) \right\rceil - 1.$$

Here  $\lceil r \rceil$  denotes, for  $r \in \mathbb{R}$ , the smallest integer not smaller than  $r$ . Then by using iteratively (2.3.44) given by Step 3, we deduce that

$$|G_{\theta, H_0}^-(x_0, t_0) \setminus A_\kappa| \leq (1 - \sigma\eta)^N |G_{\theta, H_0}^-(x_0, t_0) \setminus A_{\kappa_0}|.$$

By inserting the value of  $N$ , we conclude that

$$|G_{\theta, H_0}^-(x_0, t_0) \setminus A_\kappa| \leq \frac{1}{1 - \sigma\eta} |G_{\theta, H_0}^-(x_0, t_0)| \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon},$$

By combining (2.3.41) and (2.3.42) (see Step 2), we come back to the cube  $C_R(0, T_0)$  and obtain the desired estimate.

**Step 5** (Covering argument). We cover  $Q_{1/2}$  by

$$Q_{1/2} \left( 0, -\frac{1}{4} \right) \subseteq \bigcup_{1 \leq i \leq N} \overline{C}_R(X_i, T_i) \subseteq Q_1$$

where the parabolic cylinders  $C_R(X_i, T_i) := C_R(0, T_0) + (X_i, T_i) - (0, T_0)$  are disjoint. Then

$$|Q_{1/2} \left( 0, -\frac{1}{4} \right) \setminus A_\kappa| \leq \sum_{i=1}^N |C_R(X_i, T_i) \setminus A_\kappa|.$$

By applying Step 4 to each parabolic cylinder  $C_R(X_i, T_i)$ , we get

$$|Q_{1/2} \left( 0, -\frac{1}{4} \right) \setminus A_\kappa| \leq \frac{1}{\xi(1 - \sigma\eta)} \sum_{i=1}^N |C_R(X_i, T_i)| \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon}.$$

Since  $\sum_{i=1}^N |C_R(X_i, T_i)| \leq |Q_1|$ , we get the estimate given by (2.3.38).  $\square$



## 2.4 Proof of Theorem 2.1.2

To prove Theorem 2.1.2, we differentiate the equation to obtain the result from the parabolic  $W^{2,\varepsilon}$  estimate obtained in Section 2.3. In the elliptic case, the proof of the elliptic  $W^{3,\varepsilon}$  estimate strongly uses the  $C^{1,\alpha}$  estimates in order to apply  $W^{2,\varepsilon}$  estimates on the components of the gradient  $Du$ . In the parabolic case, the  $C^{1,\alpha}$  estimates do not imply that  $u$  is differentiable with respect to the time variable. Thus the main new challenge which arises is to upgrade the regularity with respect to time, which is accomplished in Proposition 2.4.1.

The idea of the proof of Proposition 2.4.1 is to separate time and space in order to gain local regularity. First we are going to obtain regularity in space for a fixed time  $s$ , by applying the  $W^{2,\varepsilon}$  estimate on the derivatives of  $u$ . This step gives good quadratic approximations in space and, since  $u$  solves the PDE, we obtain a first-order approximation of the solution with respect to time. Instead of obtaining directly the estimate like in the elliptic case (see [2]), we proceed by contradiction by considering a local maximum (or minimum) and we use viscosity solution arguments and the uniform ellipticity of the operator.

Note that we use  $u$  is solution of the PDE to obtain a connection between  $\Psi(u)$  and  $\Theta(u_{x_i})$ ,  $1 \leq i \leq d$ , which is different than the elliptic case.

**Proposition 2.4.1.** *Assume that  $\mathcal{F}$  satisfies (F1),  $\mathcal{F}(0) = 0$  and  $g(0, 0) = 0$ . Suppose that  $u \in C(Q_1)$  is a viscosity solution of (2.1.3) in  $Q_1$  such that  $\sup_{Q_1} |u| \leq 1$ . There exists a universal constant  $C_1 > 0$  such that, for every  $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}_+^d$  such that  $|\kappa| \geq 1$ ,*

$$\begin{aligned} \bigcap_{i=1}^d \left\{ (y, s) \in Q_{1/2} \left( 0, -\frac{1}{4} \right) : \Theta(u_{x_i}, Q_1)(y, s) \leq \kappa_i \right\} \\ \subseteq \left\{ (y, s) \in Q_{1/2} \left( 0, -\frac{1}{4} \right) : \Psi(u, Q_1)(y, s) \leq C_1 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) |\kappa| \right\}. \end{aligned}$$

*Proof.* Fix  $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}_+^d$  and  $(x_0, t_0) \in Q_{1/2} \left( 0, -\frac{1}{4} \right)$  such that, for every  $1 \leq i \leq d$ ,

$$\Theta(u_{x_i}, Q_1)(x_0, t_0) \leq \kappa_i.$$

For all  $1 \leq i \leq d$ , we select  $p^i \in \mathbb{R}^d$  such that, for every  $y \in B_1$  and  $-1 < s \leq t_0$ ,

$$|u_{x_i}(y, s) - u_{x_i}(x_0, t_0) - p^i \cdot (y - x_0)| \leq \kappa_i \left( \frac{1}{2} |x_0 - y|^2 + t_0 - s \right). \quad (2.4.1)$$

Define the polynomial approximation  $P$  to  $u$  given by

$$P(y, s) = u(x_0, t_0) + b(s - t_0) + Du(x_0, t_0) \cdot (y - x_0) + \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle \quad (2.4.2)$$

where  $b$  and  $M$  are going to be chosen. First we set  $M := (p_j^i) \in \mathbb{M}_d$ . Then to fix  $b$ , up to replacing  $M$  by  $\frac{M+M^\top}{2}$  in (2.4.2), we can assume that  $M \in \mathbb{S}_d$  and then we take

$$b := -F(M) + g(x_0, t_0). \quad (2.4.3)$$

To estimate the difference between  $P$  and  $u$ , we separate the difference into two parts, a space term and a time term

$$\begin{aligned} |u(y, s) - P(y, s)| &\leq |u(y, s) - u(x_0, s) - Du(x_0, t_0) \cdot (y - x_0) - \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle| \\ &\quad + |u(x_0, s) - u(x_0, t_0) - b(s - t_0)|. \end{aligned}$$

We proceed in eight steps. Steps 1–2 provide the upper bound on the space term. Then by introducing an adequate test function and arguing by a comparison principle argument, Steps 3–7 will give the corresponding upper bound on the time term. Finally, we will conclude in Step 8.

**Step 1.** We claim that for all  $1 \leq i \leq d$ ,  $(y, s) \in Q_1$  with  $s \leq t_0$ ,

$$|u_{x_i}(y, s) - u_{x_i}(y, t_0)| \leq \kappa_i (|x_0 - y|^2 + t_0 - s). \quad (2.4.4)$$

To prove this, notice that

$$|u_{x_i}(y, t_0) - u_{x_i}(x_0, t_0) - p^i \cdot (y - x)| \leq \frac{\kappa_i}{2} |x_0 - y|^2.$$

By combining this inequality with (2.4.1), we get

$$\begin{aligned} |u_{x_i}(y, s) - u_{x_i}(y, t_0)| &\leq |u_{x_i}(y, s) - u_{x_i}(x_0, t_0) - p^i \cdot (y - x)| + |u_{x_i}(y, t_0) - u_{x_i}(x_0, t_0) - p^i \cdot (y - x)| \\ &\leq \kappa_i \left( \frac{1}{2} |x_0 - y|^2 + t_0 - s \right) + \frac{\kappa_i}{2} |x_0 - y|^2. \end{aligned}$$

The proposed estimate (2.4.4) directly follows.

**Step 2.** We next prove the following “slice estimate”: for every  $(y, s) \in Q_1$  such that  $-1 < s \leq t_0$ , we have

$$\begin{aligned} |u(y, s) - u(x_0, s) - Du(x_0, t_0) \cdot (y - x_0) - \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle| \\ \leq |\kappa| \left( \frac{7}{6} |y - x_0|^3 + \frac{\sqrt{2}}{3} |s - t_0|^{3/2} \right). \end{aligned} \quad (2.4.5)$$

Since  $u \in C^1$  with respect to the space variable, we can write

$$\begin{aligned} I &:= \left| u(y, s) - u(x_0, s) - Du(x_0, t_0) \cdot (y - x_0) - \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle \right| \\ &= \left| (y - x_0) \cdot \int_0^1 Du(x_0 + \tau(y - x_0), s) - Du(x_0, t_0) - \tau M(x_0 - y_0) d\tau \right|. \end{aligned}$$

It is clear that  $I \leq I_1 + I_2$  where  $I_1$  and  $I_2$  respectively denote

$$I_1 := \left| (y - x_0) \cdot \int_0^1 Du(x_0 + \tau(y - x_0), s) - Du(x_0 + \tau(y - x_0), t_0) d\tau \right|$$

and

$$I_2 := \left| (y - x_0) \cdot \int_0^1 Du(x_0 + \tau(y - x_0), t_0) - Du(x_0, t_0) - \tau M(x_0 - y_0) d\tau \right|.$$

It remains to determine some upper bounds on  $I_1$  and  $I_2$ . For  $I_1$ , applying successively Cauchy-Schwarz inequality and Step 1 yields that

$$\begin{aligned} I_1 &\leq |y - x_0| \int_0^1 |Du(x_0 + \tau(y - x_0), s) - Du(x_0 + \tau(y - x_0), t_0)| d\tau \\ &\leq |y - x_0| \int_0^1 |\kappa| (\tau^2 |y - x_0|^2 + t_0 - s) d\tau \\ &\leq |\kappa| \left( \frac{1}{3} |y - x_0|^3 + |y - x_0| |s - t_0| \right). \end{aligned}$$

We estimate  $I_2$  by using (2.4.1) and the same computations than those used in [2]. For sake of completeness and reader convenience, we give here the arguments. According to (2.4.1),

$$|u_{x_i}(x_0 + \tau(y - x_0), t_0) - u_{x_i}(x_0, t_0) - \tau p^i \cdot (y - x_0)| \leq \frac{1}{2} \kappa_i \tau^2 |y - x_0|^2.$$

By Cauchy-Schwarz inequality,

$$I_2 \leq |y - x_0| \int_0^1 \frac{1}{2} \left( \sum_{i=1}^d \kappa_i^2 \right)^{1/2} \tau^2 |y - x_0|^2 d\tau = \frac{|\kappa|}{6} |y - x_0|^3.$$

To obtain (2.4.5), it suffices to apply Young inequality which yields

$$|y - x_0| |s - t_0| \leq \frac{2}{3} |y - x_0|^3 + \frac{\sqrt{2}}{3} |s - t_0|^{3/2},$$

and a simple calculation gives (2.4.5).

To complete the proof of Proposition 2.4.1, it remains now to get a similar estimate for the term in time. We will show that there exists a universal constant  $C_2 \geq 1$  such that, for all  $-1 < s' \leq t_0$ ,

$$|u(x_0, s') - u(x_0, t_0) - b(s' - t_0)| \leq C_2 \left(1 + \|g\|_{C^{0,1}(Q_1)}\right) |\kappa| |t_0 - s'|^{3/2}. \quad (2.4.6)$$

We are going to prove that, for all  $-1 < s' \leq t_0$ ,

$$u(x_0, s') - u(x_0, t_0) - b(s' - t_0) \geq -C_2 \left(1 + \|g\|_{C^{0,1}(Q_1)}\right) |\kappa| |t_0 - s'|^{3/2}, \quad (2.4.7)$$

the argument for the reverse inequality is entirely parallel. To do this, we will consider a suitable test function taking into account the size of  $b$ .

**Step 3.** We claim that the following a priori bound on  $b = -F(M) + g(x_0, t_0)$  holds

$$|b| \leq c_2 \left(|\kappa| + 1 + \|g\|_{C^{0,1}(Q_1)}\right), \quad (2.4.8)$$

where  $c_2 > 0$  a universal constant.

By considering (2.4.1) for  $s = t_0$  and selecting  $y \in \overline{B}_{3/4} \subseteq B_1$  such that  $x_0 - y = \frac{1}{4} \frac{p^i}{|p^i|}$  if  $p^i \neq 0$ , we get

$$\frac{1}{4} |p^i| - \sup_{y \in \overline{B}_{3/4}} |u_{x_i}(y, t_0) - u_{x_i}(x_0, t_0)| \leq |u_{x_i}(y, t_0) - u_{x_i}(x_0, t_0) - p^i \cdot (x_0 - y)| \leq \frac{\kappa_i}{2} |x_0 - y|^2.$$

By rearranging the terms, we obtain

$$\frac{1}{4} |p^i| \leq \frac{9}{32} \kappa_i + \sup_{y \in \overline{B}_{3/4}} |u_{x_i}(y, t_0) - u_{x_i}(x_0, t_0)| \leq \frac{9}{32} \kappa_i + 2 \sup_{\overline{Q}_{3/4}} |Du|.$$

Now by Proposition 2.2.1, there exists a universal constant  $\tilde{C} > 0$  such that

$$\sup_{\overline{Q}_{3/4}} |Du| \leq \tilde{C} \left(1 + \|g\|_{C^{0,1}(Q_1)}\right).$$

We deduce that there exists a universal constant  $c > 9/8$  such that, for all  $1 \leq i \leq d$ ,

$$|p^i| \leq c \left(\kappa_i + 1 + \|g\|_{C^{0,1}(Q_1)}\right).$$

If  $\|M\|$  denotes the  $(L^2, L^2)$ -norm of  $M$ , i.e.  $\|M\| = \sup_{|x|=1} |Mx|$ , then our choice for  $M = (p_j^i)$  implies that

$$\|M\| \leq dc \left(|\kappa| + 1 + \|g\|_{C^{0,1}(Q_1)}\right).$$

Since  $\mathcal{F}$  satisfies (F1) and  $\mathcal{F}(0) = 0$ , it is immediate to check that for all  $N \in \mathbb{S}_d$ ,  $|\mathcal{F}(N)| \leq d\Lambda \|N\|$ . Thus, by using (2.4.3),  $g(0, 0) = 0$  and  $(x_0, t_0) \in Q_1$ , we get

$$|b| \leq d\Lambda \|M\| + |g(x_0, t_0)| \leq d\Lambda \|M\| + \|g\|_{C^{0,1}(Q_1)},$$

and the claim (2.4.8) easily follows.

**Step 4.** Let  $s' \in (-1, t_0]$ . Next we show that

$$\phi \geq u \quad \text{holds on } \partial_p (B_{3/4} \times \{s' < s \leq t_0\}),$$

where we have defined

$$\begin{aligned} \phi(y, s) &:= u(x_0, s') + Du(x_0, t_0) \cdot (y - x_0) + \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle \\ &\quad + \left(b + \left(\frac{C_0 \beta}{4} + 2 \|g\|_{C^{0,1}(Q_1)}\right) |\kappa| |s' - t_0|^{1/2}\right) (s - s') + \frac{C_0}{6} |\kappa| (|y - x_0|^3 + |s' - t_0|^{3/2}) \end{aligned}$$

where  $\beta := \max\{1, 2\Lambda(d+1)\}$  and  $C_0$  is a constant depending on  $d, \lambda, \Lambda$  and  $\|g\|_{C^{0,1}(Q_1)}$  given by

$$C_0 := 768 \left( c_2 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) + 3 \right). \quad (2.4.9)$$

First we claim that  $\phi \geq u$  on  $\overline{B}_{3/4} \times \{s'\}$ . To prove this, it follows from (2.4.5) that for all  $y \in \overline{B}_{3/4} \subseteq B_1$ ,

$$u(y, s') \leq u(x_0, s') + Du(x_0, t_0) \cdot (y - x_0) + \frac{1}{2} \langle (y - x_0), M(y - x_0) \rangle + |\kappa| \left( \frac{7}{6} |y - x_0|^3 + \frac{\sqrt{2}}{3} |s' - t_0|^{3/2} \right).$$

Since  $\frac{C_0}{6} \geq \max\left\{\frac{7}{6}, \frac{\sqrt{2}}{3}\right\}$ , this yields that

$$u(\cdot, s') \leq \phi(\cdot, s') \quad \text{on } \overline{B}_{3/4}.$$

Then we claim that  $\phi \geq u$  on  $\{(y, s) : |y| = 3/4, s' < s \leq t_0\}$ . Arguing by contradiction, assume that there exists  $(x_1, t_1)$  with  $|x_1| = 3/4, s' < t_1 \leq t_0$  such that

$$\phi(x_1, t_1) < u(x_1, t_1).$$

Then,

$$\phi(x_1, t_1) - u(x_1, s') < |u(x_1, t_1) - u(x_1, s')| \leq 2.$$

Moreover,

$$\begin{aligned} \phi(x_1, t_1) - u(x_1, s') &= \frac{C_0}{6} |\kappa| \left( |x_1 - x_0|^3 + |s' - t_0|^{3/2} \right) \\ &\quad + u(x_0, s') - u(x_1, s') + Du(x_0, t_0) \cdot (x_1 - x_0) + \frac{1}{2} \langle (x_1 - x_0), M(x_1 - x_0) \rangle \\ &\quad + \left( b + \left( 2 \|g\|_{C^{0,1}(Q_1)} + \frac{C_0}{4} \beta \right) |\kappa| |s' - t_0|^{1/2} \right) \underbrace{(t_1 - s')}_{\geq 0}. \end{aligned}$$

Neglecting the nonnegative time terms, we obtain

$$\begin{aligned} \phi(x_1, t_1) - u(x_1, s') &\geq \frac{C_0}{6} |\kappa| |x_1 - x_0|^3 - |b| |t_1 - s'| \\ &\quad - |u(x_0, s') - u(x_1, s') + Du(x_0, t_0) \cdot (x_1 - x_0) + \frac{1}{2} \langle (x_1 - x_0), M(x_1 - x_0) \rangle|. \end{aligned}$$

By combining the slice estimate (2.4.5) and the bound on  $b$  given by (2.4.8), we obtain that

$$\begin{aligned} \phi(x_1, t_1) - u(x_1, s') &\geq \frac{C_0}{6} |\kappa| |x_1 - x_0|^3 \\ &\quad - c_2 \left( |\kappa| + 1 + \|g\|_{C^{0,1}(Q_1)} \right) |t_1 - s'| - |\kappa| \left( \frac{7}{6} |x_1 - x_0|^3 + \frac{\sqrt{2}}{3} |t_1 - t_0|^{3/2} \right). \end{aligned}$$

Using that  $\frac{1}{4} \leq |x_1 - x_0| \leq \frac{5}{4}$ ,  $|t_1 - t_0| \leq 1$  and  $|t_1 - s'| \leq 1$ , the inequality above simply reduces to

$$\frac{C_0}{384} |\kappa| < \left( c_2 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) + 3 \right) (|\kappa| + 1).$$

By inserting (2.4.9), we obtain  $2|\kappa| < |\kappa| + 1$ , and we get a contradiction if  $|\kappa| \geq 1$ .

**Step 5.** We claim that:

$$u - \phi \text{ can attain a positive global maximum only in } \left\{ (y, s) : |y - x_0| < |s - t_0|^{1/2} \right\}.$$

By Step 4, we may assume that  $u - \phi$  attains a positive global maximum at  $(x_1, t_1) \in B_{3/4} \times \{s' < s \leq t_0\}$ :

$$(u - \phi)(x_1, t_1) = \sup_{B_{3/4} \times \{s' < s \leq t_0\}} (u - \phi)(y, s).$$

In particular,

$$(u - \phi)(x_1, t_1) \geq (u - \phi)(x_0, t_1).$$

A direct computation yields

$$\phi(x_1, t_1) - \phi(x_0, t_1) = Du(x_0, t_0) \cdot (x_1 - x_0) + \frac{1}{2} \langle (x_1 - x_0), M(x_1 - x_0) \rangle + \frac{C_0}{6} |\kappa| |x_1 - x_0|^3.$$

After rearranging the terms, we obtain

$$u(x_1, t_1) - u(x_0, t_1) - Du(x_0, t_0) \cdot (x_1 - x_0) - \frac{1}{2} \langle (x_1 - x_0), M(x_1 - x_0) \rangle \geq \frac{C_0}{6} |\kappa| |x_1 - x_0|^3. \quad (2.4.10)$$

Moreover, we know by (2.4.5) that

$$\begin{aligned} |u(x_1, t_1) - u(x_0, t_1) - Du(x_0, t_0) \cdot (x_1 - x_0) - \frac{1}{2} \langle (x_1 - x_0), M(x_1 - x_0) \rangle| \\ \leq |\kappa| \left( \frac{7}{6} |x_1 - x_0|^3 + \frac{\sqrt{2}}{3} |t_1 - t_0|^{3/2} \right). \end{aligned}$$

By using (2.4.10) we deduce from the inequality above that

$$|\kappa| \left( \frac{7}{6} |x_1 - x_0|^3 + \frac{\sqrt{2}}{3} |t_1 - t_0|^{3/2} \right) \geq \frac{C_0}{6} |\kappa| |x_1 - x_0|^3.$$

Since  $\frac{C_0}{6} \geq \frac{7}{3}$ , we obtain

$$|x_1 - x_0| \leq \left( \frac{2\sqrt{2}}{7} \right)^{1/3} |t_1 - t_0|^{1/2}.$$

In particular, this yields the desired claim.

**Step 6.** We next show that  $u - \phi$  cannot achieve any local maximum in the cylinder  $\tilde{Q}$  given by

$$\tilde{Q} := \{(y, s) \in B_1 \times [s', t_0] : |y - x_0| < |s' - t_0|^{1/2}\},$$

by arguing that  $\phi$  is a strict supersolution in  $\tilde{Q}$  i.e.,

$$\partial_t \phi + F(D^2 \phi) > g, \quad \text{in } \tilde{Q}. \quad (2.4.11)$$

We verify (2.4.11) by a direct computation. By the uniform ellipticity condition (F1), and noticing that the perturbation is a positive matrix,

$$F \left( M + \frac{C_0 |\kappa|}{2} |y - x_0| \left[ I_d + \frac{y - x_0}{|y - x_0|} \otimes \frac{y - x_0}{|y - x_0|} \right] \right) \geq F(M) - \frac{\Lambda(d+1)}{2} C_0 |\kappa| |y - x_0|.$$

This inequality yields

$$\begin{aligned} b + \left( 2 \|g\|_{C^{0,1}(Q_1)} + \frac{\beta}{4} C_0 \right) |\kappa| |s' - t_0|^{1/2} + F \left( M + \frac{C_0 |\kappa|}{2} |y - x_0| \left[ I_d + \frac{y - x_0}{|y - x_0|} \otimes \frac{y - x_0}{|y - x_0|} \right] \right) \\ \geq b + 2 \|g\|_{C^{0,1}(Q_1)} |\kappa| |s' - t_0|^{1/2} + F(M) + \frac{\beta}{4} C_0 |\kappa| |s' - t_0|^{1/2} - \frac{\Lambda(d+1)}{2} C_0 |\kappa| |y - x_0|. \end{aligned}$$

For our choice given by (2.4.3), we have  $b + F(M) = g(x_0, t_0)$ . Recalling the value of  $\beta$  and  $|\kappa| \geq 1$ , we obtain that the following inequality holds in  $\tilde{Q}$ :

$$\begin{aligned} b + \left( 2 \|g\|_{C^{0,1}(Q_1)} + \frac{\beta}{4} C_0 \right) |\kappa| |s' - t_0|^{1/2} + F \left( M + \frac{C_0 |\kappa|}{2} |y - x_0| \left[ I_d + \frac{y - x_0}{|y - x_0|} \otimes \frac{y - x_0}{|y - x_0|} \right] \right) \\ > \|g\|_{C^{0,1}(Q_1)} (|y - x_0| + |s - t_0|^{1/2}) + g(x_0, t_0) \geq g(y, s). \end{aligned}$$

This confirms (2.4.11).

**Step 7.** By Steps 4–6,  $u - \phi$  cannot achieve any positive global maximum. Hence,

$$u \leq \phi \quad \text{in } \overline{B}_{3/4} \times [s', t_0].$$

In particular, for  $s = t_0$ ,  $u(x_0, t_0) \leq \phi(x_0, t_0)$ , this yields

$$u(x_0, t_0) \leq u(x_0, s') + b(t_0 - s') + \left( \frac{C_0 \beta}{4} + 2 \|g\|_{C^{0,1}(Q_1)} \right) |\kappa| |t_0 - s'|^{3/2} + \frac{C_0}{6} |\kappa| |t_0 - s'|^{3/2}.$$

After rearranging the terms, we get

$$- \left( C_0 \left( \frac{1+\beta}{4} \right) + 2 \|g\|_{C^{0,1}(Q_1)} \right) |\kappa| |t_0 - s'|^{3/2} \leq u(x_0, s') - u(x_0, t_0) - b(s' - t_0).$$

Thus, by using (2.4.9), it follows (2.4.7) by taking  $C_2 := 192(1 + \beta)(c_2 + 3)$ .

**Step 8.** We conclude the argument. By combining both (2.4.5) and (2.4.6), we deduce that, for all  $(y, s) \in Q_1$ ,  $s \leq t_0$ ,

$$\begin{aligned} |u(y, s) - P(y, s)| &\leq |\kappa| \left( \frac{7}{6} |y - x_0|^3 + \frac{\sqrt{2}}{3} |s - t_0|^{3/2} \right) + C_2 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) |\kappa| |t_0 - s|^{3/2} \\ &\leq \left( C_2 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) + \frac{7}{6} \right) |\kappa| \left( |y - x_0|^3 + |s - t_0|^{3/2} \right). \end{aligned}$$

By setting  $C_1 := C_2 + \frac{7}{6}$ , this implies  $\Psi(u, Q_1)(x_0, t_0) \leq C_1 \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right) |\kappa|$ .  $\square$

Now we can give the proof of the parabolic  $W^{3,\varepsilon}$  estimate stated in the introduction.

*Proof of Theorem 2.1.2.* If  $u \equiv 0$  on  $Q_1$ , then the estimate is clear, so we may assume that  $\sup_{Q_1} |u| > 0$ . In Step 1, we reduce the proof of the theorem to the case of  $\sup_{Q_1} |u| \leq 1$ ,  $g(0, 0) = 0$  and  $\mathcal{F}(0) = 0$  by scaling arguments. In Step 2, we prove the theorem under these assumptions.

**Step 1.** We first reduce to the case that  $g(0, 0) = 0$ . If  $g(0, 0) \neq 0$ , define  $\bar{g}(x, t) := g(x, t) - g(0, 0)$  and  $\bar{u}(x, t) := u(x, t) - tg(0, 0)$  is a solution of

$$\partial_t \bar{u} + \mathcal{F}(D^2 \bar{u}) = \bar{g}.$$

By direct computation,

$$\sup_{Q_1} |\bar{u}| \leq \sup_{Q_1} |u| + \sup_{-1 \leq t \leq 0} |t| |g(0, 0)| \leq \sup_{Q_1} |u| + |g(0, 0)|.$$

Then we reduce to the case that  $\mathcal{F}(0) = 0$ . If  $\mathcal{F}(0) \neq 0$ , then, by ellipticity, there exists  $a \in \mathbb{R}$  such that

$$\mathcal{F}(aI_d) = 0 \quad \text{with } |a| \leq \frac{1}{\lambda d} |\mathcal{F}(0)|. \quad (2.4.12)$$

Define the operator  $\hat{\mathcal{F}}(M) := \mathcal{F}(M + aI_d)$  and observe that  $\hat{\mathcal{F}}$  satisfies (F1) with the same ellipticity constants  $\lambda, \Lambda$  and  $\hat{\mathcal{F}}(0) = \mathcal{F}(aI_d) = 0$ . It is clear that  $\hat{u}(x, t) := u(x, t) - \frac{1}{2}a|x|^2$  is a solution of

$$\partial_t \hat{u} + \hat{\mathcal{F}}(D^2 \hat{u}) = 0$$

By direct computation,

$$\sup_{Q_1} |\hat{u}| \leq \sup_{Q_1} |u| + \frac{1}{2}|a| \sup_{x \in B_1} |x|^2 \leq \sup_{Q_1} |u| + \frac{1}{2}|a|.$$

By applying (2.4.14) and using (2.4.12), we obtain

$$\left| \left\{ (x, t) \in Q_{1/2} \left( 0, -\frac{1}{4} \right) : \Psi(u, Q_{3/4})(x, t) > \kappa \right\} \right| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u| + \frac{1}{2\lambda d} |\mathcal{F}(0)|} \right)^{-\varepsilon},$$

and we get the inequality given by Theorem 2.1.2.

Next we reduce to the case that  $\sup_{Q_1} |u| \leq 1$ . Assume that we have shown if  $\sup_{Q_1} |u| \leq 1$  and  $\mathcal{F}(0) = 0$ , then for all  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(u, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon}. \quad (2.4.13)$$

We claim that if  $\mathcal{F}(0) = 0$ , and  $\beta := \sup_{Q_1} |u| > 0$  then for all  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(u, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\beta + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon}. \quad (2.4.14)$$

Define the function  $\tilde{g} := (1/\beta)g$ , the operator  $\tilde{\mathcal{F}}(M) := \beta^{-1}\mathcal{F}(\beta M)$  and observe  $\tilde{g} \in C^{0,1}(Q_1)$  with  $\|\tilde{g}\|_{C^{0,1}(Q_1)} = \|g\|_{C^{0,1}(Q_1)}/\beta$ ,  $\tilde{\mathcal{F}}$  satisfies (F1) with the same ellipticity constants  $\lambda, \Lambda$  and  $\tilde{\mathcal{F}}(0) = \beta^{-1}\mathcal{F}(0) = 0$ . It is clear that  $\tilde{u} := u/\beta$  is a solution of

$$\partial_t \tilde{u} + \tilde{\mathcal{F}}(D^2 \tilde{u}) = \tilde{g}$$

with  $\sup_{Q_1} |\tilde{u}| = 1$ . By applying (2.4.13) to  $\tilde{u}$ , we obtain that, for all  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(\tilde{u}, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(Q_1)}/\beta} \right)^{-\varepsilon}.$$

Noticing that  $\Psi(\tilde{u}, Q_{3/4})(x, t) = \frac{1}{\beta}\Psi(u, Q_{3/4})(x, t)$ , we obtain that, for all  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(u, Q_{3/4})(x, t) > \beta\kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(Q_1)}/\beta} \right)^{-\varepsilon}.$$

This is equivalent to (2.4.14).

**Step 2.** Assuming that  $\sup_{Q_1} |u| \leq 1$ ,  $g(0, 0) = 0$  and  $\mathcal{F}(0) = 0$ , we give the proof of (2.4.13). It suffices to get the inequality for  $\kappa \geq \kappa_1$ , where  $\kappa_1$  is a universal constant. According to Proposition 2.2.1,  $Du$  is continuous and there exists a constant  $C$  such that

$$\sup_{Q_{3/4}} |Du| \leq C \left( 1 + \|g\|_{C^{0,1}(Q_1)} \right). \quad (2.4.15)$$

Moreover, we claim that for every unit direction  $e \in \mathbb{R}^d$ ,  $|e| = 1$ , the function  $u_e = e \cdot Du$  satisfies the inequalities

$$\partial_t u_e + \mathcal{P}_{\lambda, \Lambda}^-(u_e) - \|g\|_{C^{0,1}(Q_1)} \leq 0 \leq \partial_t u_e + \mathcal{P}_{\lambda, \Lambda}^+(u_e) + \|g\|_{C^{0,1}(Q_1)}, \quad \text{in } Q_1,$$

in the viscosity sense. We refer to [34, Lemma 3.12] for the elliptic version of this statement which is easy to generalize to the parabolic setting.

According to Proposition 2.3.1, we have, for each  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u_e, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{Q_{3/4}} |u_e| + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon},$$

where  $C, \varepsilon > 0$  are universal constants. Thus, we deduce from (2.4.15) that there exists a new universal constant  $C > 0$  such that for all  $\kappa > 0$ ,

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \underline{\Theta}(u_e, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(Q_1)}} \right)^{-\varepsilon}. \quad (2.4.16)$$

By Proposition 2.4.1, there exists a universal constant  $C_1$  such that, for all  $\kappa > C_1 \left(1 + \|g\|_{C^{0,1}(Q_1)}\right)$ ,

$$\begin{aligned} & |\{(x, t) \in Q_{1/2} \left(0, -\frac{1}{4}\right) : \Psi(u, Q_{3/4})(x, t) > \kappa\}| \\ & \leq \sum_{i=1}^d \left| \left\{ (x, t) \in Q_{1/2} \left(0, -\frac{1}{4}\right) : \Theta(u_{x_i}, Q_{3/4})(x, t) > \frac{\kappa}{C_1 \left(1 + \|g\|_{C^{0,1}(Q_1)}\right) \sqrt{d}} \right\} \right|, \end{aligned}$$

and we obtain the desired result for  $\kappa > C_1 \left(1 + \|g\|_{C^{0,1}(Q_1)}\right)$  by applying (2.4.16). This completes the proof of (2.4.13).  $\square$





# Rates of convergence for dynamic programming schemes to fully nonlinear, uniformly elliptic equations

Nous étudions le taux de convergence pour les solutions de schémas correspondant à la famille de jeux répétés à deux personnes dépendant d'un petit paramètre  $\varepsilon$  proposés par Kohn et Serfaty. La solution du schéma résout une équation de programmation dynamique et converge vers la solution de viscosité de l'équation elliptique lorsque le paramètre  $\varepsilon$  tend vers zéro. Nous nous plaçons sous une hypothèse d'uniforme ellipticité et nous traitons le cas d'une condition au bord de Dirichlet.

Preprint
----------



## Chapter 3

# Rates of convergence for dynamic programming schemes to fully nonlinear, uniformly elliptic equations

### Abstract

We give an algebraic rate of convergence for the schemes given by the scores associated to the family of two-person games depending on a small parameter  $\varepsilon$  proposed by Kohn and Serfaty in a smooth bounded domain. The solution of the scheme solves a dynamic programming equation and converges, as the small parameter tends to zero, to the viscosity solution of an elliptic PDE with a Dirichlet boundary condition. Our rate requires a uniform ellipticity assumption on the nonlinear operator which depends only  $D^2u$ . Our method mainly relies on an elliptic  $W^{3,\sigma}$  estimate and suitable Aleksandrov-Bakelman-Pucci inequalities.

**Keywords:** rate of convergence, fully nonlinear elliptic equations, viscosity solutions.

**Mathematics Subject Classification** 35B65, 35J60, 35D40, 65M15.

### Contents

---

<b>3.1</b>	<b>Introduction</b>	<b>140</b>
3.1.1	Motivation and statement of the result	140
3.1.2	Hypotheses and notation	141
<b>3.2</b>	<b>Proof of the rate of convergence</b>	<b>144</b>
3.2.1	Rate near the boundary	144
3.2.2	Rate far the boundary	145
<b>3.3</b>	<b>The dynamic programming schemes : presentation and properties</b>	<b>148</b>
3.3.1	Presentation of the games	148
3.3.2	Proof of the assumptions (S1) and (S2) on $F_\varepsilon$	150
3.3.3	Proof of Proposition 3.1.2	151
<b>3.4</b>	<b>Proof of Proposition 3.2.2</b>	<b>161</b>
<b>3.5</b>	<b>Appendix</b>	<b>166</b>

---

## 3.1 Introduction

### 3.1.1 Motivation and statement of the result

We consider the fully nonlinear elliptic equation

$$\begin{cases} \mathcal{F}(D^2u) + \xi u = 0, & \text{in } U \subseteq \mathbb{R}^d, \\ u = g, & \text{on } \partial U, \end{cases} \quad (3.1.1)$$

where  $\xi > 0$  and the operator  $\mathcal{F} : \mathcal{S}_d \rightarrow \mathbb{R}$  is uniformly elliptic with ellipticity constants  $\lambda$  and  $\Lambda$  such that  $0 < \lambda \leq \Lambda$ ,  $\mathcal{S}_d$  is the set of  $d$ -by- $d$  real symmetric matrices and  $D^2\phi \in \mathcal{S}_d$  denotes the Hessian matrix of a function  $\phi$ .

In [88], Kohn and Serfaty proposed a deterministic control interpretation, via a two persons repeated games, of a broad class of fully nonlinear elliptic or parabolic type. Their games have two opposing players, Helen and Mark, who always make decisions rationally and deterministically. The rules depend on the form of the equation, but there is always a small parameter  $\varepsilon$ , which governs the spatial step size. Helen's goal is to optimize her worst-case outcome. The value function  $u_\varepsilon$  of Helen can be characterized by a dynamic programming principle. Kohn and Serfaty showed that  $u_\varepsilon$  converges locally uniformly to the viscosity solution of (3.1.1) as  $\varepsilon$  goes to 0. Therefore, the game interpretation gives a scheme to approximate the viscosity solution of the PDE. An open question raised in this article is to determine an algebraic rate of convergence for these schemes associated to the games.

We answer here the question on the elliptic case setting and the Dirichlet problem, solving (3.1.1) in a smooth domain  $U$  with  $u = g$  at  $\partial U$ . For the sake of simplicity we reduce the study to the case when the nonlinearity  $\mathcal{F}$  depends only on the Hessian. In particular, the PDE has a comparison principle and the constant  $\xi$  in (3.1.1) which plays the role of an interest rate in the game has no additional condition since  $\mathcal{F}$  is independent of the variable  $u$ . In that case, the dynamic programming equation satisfied by the value function  $u_\varepsilon$  is given by

$$u_\varepsilon(x) = S_\varepsilon[x, u_\varepsilon] \quad \text{on } U, \quad (3.1.2)$$

where  $S_\varepsilon$  is the dynamic programming operator associated to the game given by (3.1.8). It is important to notice that the boundary condition is already taken into account in the operator  $S_\varepsilon$ .

The study of the convergence and error estimates for monotone and consistent approximations to fully nonlinear, first-order PDEs were established by Crandall and Lions [43] and Souganidis [133]. The convergence, without error, of monotone and consistent approximations for fully nonlinear, possibly degenerate second-order equations PDEs was first obtained by Barles and Souganidis [20]. For finite difference schemes Kuo and Trudinger [98, 99] also investigated in great detail at the issues of regularity and existence of such approximations for uniformly elliptic equations. Finding a rate of convergence has been the lack of appropriate regularizations of viscosity solutions yielding control on derivatives higher than 2 except for either convex or concave  $\mathcal{F}$ .

The first result in the convex/concave case, with  $h = \frac{1}{27}$ , was obtained by Krylov [93, 94] using the stochastic control interpretation of the equation that is available in the convex/concave setting. Later Barles and Jakobsen [15, 16] improved the error to  $h = \frac{1}{5}$  by purely PDEs techniques using switching-control-type approximations – once again the convexity/concavity of the operator  $\mathcal{F}$  plays a crucial role. More recently, Krylov [95], always in the convex/concave but degenerate case, improved the rate to  $h = \frac{1}{2}$  again using stochastic control considerations.

The main difficulty to get an error estimate between viscosity solution and approximate solutions, in both the elliptic and parabolic settings, is overcoming the lack of regularity of the viscosity solution  $u$ . The proof of the error estimate in [35] is based on the  $W^{3,\varepsilon}$  estimate, which says that outside a set of

small measure, the solution of a uniformly elliptic equation has second-order expansions with controlled cubic term. In their article, Caffarelli and Souganidis strongly use the notion of inf-, sup- convolution introduced by Jensen [86]. This error estimate was also a key step in getting a rate for homogenization in random media [36].

In this paper, we aim to give an alternative strategy for studying the rate of convergence of non divergence form equations that we believe to be of interest beyond the proof of the rate of convergence associated to (3.1.1). Here we propose a new proof based on ABP inequalities and a measure argument. We study the error by examining the maximum points of a family functions on doubled variables which are indexed by some small slopes. A key point of the argument relies on the convex envelop and the Aleksandrov-Bakelman-Pucci (ABP) inequality. The other tool is the regularity theorem [2].

Now we can state our main result. The hypotheses are given in the next subsection.

**Theorem 3.1.1** (Rate of convergence, D.). *Let  $U$  be a  $C^2$ -bounded domain. Assume that  $\mathcal{F}$  and  $F_\varepsilon$  satisfy respectively (F1)-(F2) and (S1)-(S2). Let  $u$  and  $u_\varepsilon$  be respectively the viscosity solution of (3.1.1) and the solution of (3.1.2). Then there exist some constants  $\eta$  and  $C$  depending on  $\lambda$ ,  $\Lambda$ ,  $d$  and  $\|g\|_{C^{0,1}(\partial U)}$  such that*

$$\sup_U |u - u_\varepsilon| \leq C\varepsilon^\eta.$$

**Outline of the paper.** In Section 3.1.2, we give the notation, present the scheme corresponding to the game and list the assumptions on both the nonlinearity and the approximate operator. the Section 3.2, we give the proof of Theorem 3.1.1 and state the two tools required. Then Section 3.3 gives a detailed presentation of the schemes and show the properties (S1), (S2) and Proposition 3.1.2. Finally, Section 3.4 is devoted to the proof of Proposition 3.2.2.

### 3.1.2 Hypotheses and notation

Throughout the paper, we work in  $\mathbb{R}^d$  in dimension  $d \geq 2$  and all the differential equations and inequalities are to be interpreted in the viscosity sense (c.f. [34, 45]).  $\mathcal{S}_d$  denotes the set of symmetric  $d \times d$  matrices. If  $x, y \in \mathbb{R}^d$ ,  $\langle x, y \rangle$  denotes the usual Euclidean inner product and  $\|x\|$  the Euclidean length of  $x$ . If  $A$  is a  $d \times d$  matrix,  $\|A\|$  denotes the operator norm  $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$ .

Given  $0 < \lambda \leq \Lambda$ , the Pucci extremal operators  $\mathcal{P}_{\lambda, \Lambda}^+$  and  $\mathcal{P}_{\lambda, \Lambda}^-$  are defined for each  $A \in \mathcal{S}_d$  by

$$\mathcal{P}_{\lambda, \Lambda}^+(M) = -\lambda \operatorname{tr}(M_+) + \Lambda \operatorname{tr}(M_-) \quad \text{and} \quad \mathcal{P}_{\lambda, \Lambda}^-(M) = -\Lambda \operatorname{tr}(M_+) + \lambda \operatorname{tr}(M_-), \quad (3.1.3)$$

Here  $\operatorname{tr}(M)$  denotes the trace of  $M$ , and  $M_+, M_-$  are the uniquely defined matrices by the conditions :  $M = M_+ - M_-$ ,  $M_+ M_- = 0$  and  $M_+, M_- \geq 0$ . The identity matrix is denoted by  $I_d$ . where  $\mu_1, \dots, \mu_d$  are the eigenvalues of  $M$ . These operators satisfy the inequalities

$$\begin{aligned} \mathcal{P}_{\lambda, \Lambda}^-(M) + \mathcal{P}_{\lambda, \Lambda}^-(N) &\leq \mathcal{P}_{\lambda, \Lambda}^-(M + N) \leq \mathcal{P}_{\lambda, \Lambda}^-(M) + \mathcal{P}_{\lambda, \Lambda}^+(N) \\ &\leq \mathcal{P}_{\lambda, \Lambda}^+(M + N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M) + \mathcal{P}_{\lambda, \Lambda}^+(N). \end{aligned} \quad (3.1.4)$$

The term domain will be reserved for a nonempty, connected, and open subset of  $\mathbb{R}^d$ . Throughout this article,  $U$  will denote a smooth ( $C^2$ ) bounded domain. In particular, it satisfies the two following geometric conditions. The domain  $U$  satisfies the *uniform interior ball condition* if there exists  $r > 0$  such that for all  $x \in \partial U$  there exists an open ball  $B \subset U$  with  $x \in \partial B$  and radius  $r$ . The domain  $U$  satisfies the *uniform exterior ball condition* if  $\mathbb{R}^d \setminus \overline{U}$  satisfies the uniform interior ball condition. We denote path distance on  $\overline{U}$  by  $\operatorname{dist}$ . That is,  $\operatorname{dist}(x, y)$  is the infimum of the lengths of all possible Lipschitz paths  $\gamma: [0, 1] \rightarrow \overline{U}$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . We denote by  $\operatorname{diam} U$  the diameter of  $U$  with respect to  $d$ ; i.e.,

$$\operatorname{diam} U = \sup\{\operatorname{dist}(x, y) : x, y \in \overline{U}\}.$$

The regularity of the domain  $U$  gives that  $\text{diam } U < +\infty$ . In addition, we define

$$U(r) := \{x \in \overline{U} : \text{dist}(x, \partial U) \leq r\}. \quad (3.1.5)$$

If  $K$  is a compact subset of  $\overline{U}$  and  $h: K \rightarrow \mathbb{R}$  is continuous, we define the modulus of  $h$  on  $K$  by

$$\omega_h(s) := \max\{t|h(x) - h(y)| : t|x - y| \leq s \text{ and } t \geq 1\}, s \geq 0.$$

It is easy to check that  $|h(x) - h(y)| \leq \omega_h(|x - y|) \leq \omega_h(\text{dist}(x, y))$  and that  $\omega_h$  is continuous, nondecreasing, concave on  $[0, \infty)$  and  $\omega_h(0) = 0$ . In particular,

$$\omega_h(ts) \leq t\omega_h(s) \quad \text{for every } s \geq 0, t \geq 1.$$

We call any function  $\omega$  with the properties above a *modulus of continuity* for  $h$ .

The convex envelope of a function  $u \in C(U)$  and  $x \in U$  is denoted by

$$\Gamma_u(x) := \sup_{p \in \mathbb{R}^d} \inf_{y \in U} (u(y) + p \cdot (x - y)).$$

Given a function  $w \in C(U)$  and  $x \in U$ , the *subdifferential* of  $w$  at  $x$  is denoted by

$$\partial w(x) := \{p \in \mathbb{R}^d : w(y) \geq w(x) + p \cdot (y - x) \text{ for all } y \in U\}.$$

and, for each  $V \subseteq U$ , we denote the image of  $V$  under  $\partial w$  by

$$\partial w(V) := \bigcup_{x \in V} \partial w(x).$$

We now present the schemes involved in this chapter. They are derived through a dynamic programming principle. More precisely, we need to introduce three parameters  $\alpha, \beta, \gamma > 0$ , whose presence represents no loss of generality. The requirements

$$\alpha < 1/3 \quad (3.1.6)$$

and

$$\alpha + \beta < 1, \quad 2\alpha + \gamma < 2. \quad (3.1.7)$$

For the game corresponding to the stationary equation, we consider the dynamic programming operator  $S_\varepsilon$  defined for any  $x \in U$ ,  $z \in \mathbb{R}$ , and any continuous function  $\phi: U \rightarrow \mathbb{R}$ , by

$$S_\varepsilon[x, \phi] = \sup_{p, \Gamma} \inf_{\Delta x} \left[ e^{-\xi \varepsilon^2} \hat{\phi}(x + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \right], \quad (3.1.8)$$

with the restrictions that  $p$ ,  $\Gamma$  and  $\Delta x$  are constrained by

$$\|p\| \leq \varepsilon^{-\beta} \quad \text{and} \quad \|\Gamma\| \leq \varepsilon^{-\gamma} \quad (3.1.9)$$

and

$$\|\Delta x\| \leq \varepsilon^{1-\alpha}, \quad (3.1.10)$$

and the convention that, for  $\phi \in C(U)$ , denote  $\hat{\phi}$  is the function defined by

$$\hat{\phi}(x) = \begin{cases} \phi(x), & \text{if } x \in U, \\ g(x), & \text{if } x \notin U. \end{cases}$$

In particular, this operator is monotone, i.e. if  $\phi_1 \leq \phi_2$ , then  $S_\varepsilon[x, \phi_1] \leq S_\varepsilon[x, \phi_2]$ . The dynamic programming principle can be concisely written in the form (3.1.2). The statement of the convergence

theorem proved by Kohn and Serfaty is given Section 3.3.1. Now we introduce the operator  $F_\varepsilon$  which is consistent with the nonlinearity  $\mathcal{F}$ . More precisely,  $F_\varepsilon$  is defined for  $x \in U$  and  $u \in L^\infty(U)$  by

$$F_\varepsilon[u](x) := \frac{1}{\varepsilon^2} (u(x) - S_\varepsilon[x, u]). \quad (3.1.11)$$

where the operator  $S_\varepsilon$  is defined by (3.1.8). By observing that the action of the operator  $S_\varepsilon$  at a point  $x \in U$  on a function  $\phi$  requires the knowledge of  $\phi$  on  $B(x, \varepsilon^{1-\alpha})$ , we are led to introduce the notions of approximate adherence and boundary associated to the scheme. More precisely, for  $H \subseteq U$ , we define the approximate adherence of  $H$  by

$$\text{Adh}_\varepsilon H := \{x \in \mathbb{R}^d : \text{dist}(x, H) \leq \varepsilon^{1-\alpha}\}$$

and the approximate boundary of  $H$  by

$$\partial_\varepsilon H := \text{Adh}_\varepsilon H \setminus H. \quad (3.1.12)$$

Finally we list our hypotheses concerning the nonlinearity  $\mathcal{F}$  and the approximate operator  $F_\varepsilon$ .

- (F1) **(Uniform ellipticity)** The nonlinear elliptic operator  $\mathcal{F} : \mathcal{S}_d \rightarrow \mathbb{R}$  is assumed to be uniformly elliptic and Lipschitz; precisely, we assume that there exist constants  $0 < \lambda \leq \Lambda$  such that, for every  $M, N \in \mathcal{S}_d$ ,

$$\mathcal{P}_{\lambda, \Lambda}^-(M - N) \leq \mathcal{F}(M) - \mathcal{F}(N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M - N).$$

- (F2)  $\mathcal{F}(0) = 0$ .

The approximate operator  $F_\varepsilon$  satisfies the two assumptions:

- (S1) **(Comparison principle)** Assume that  $u, -v \in \text{LSC}(U)$  are bounded and satisfy

$$F_\varepsilon[u] \leq F_\varepsilon[v] \quad \text{on } H \subseteq U.$$

Then

$$\sup_H (u - v) \leq e^{-\xi \varepsilon^2} \max_{\partial_\varepsilon H} (u - v),$$

where  $\partial_\varepsilon H$  is defined by (3.1.12).

- (S2) **(Consistency)** There exists a universal constant  $C > 0$  such that for any  $x \in U$  such that  $\text{dist}(x, \partial U) > \varepsilon^{1-\alpha}$  and any smooth function  $\phi$  defined near  $x$ ,  $F_\varepsilon$  being defined by (3.1.8), we have

$$|F_\varepsilon[x, \phi] - (\mathcal{F}(D^2\phi(x)) + \xi\phi(x))| \leq C \left( 1 + \sup_{B(x, \varepsilon^{1-\alpha})} \|D^3\phi\| \right) \varepsilon^\nu$$

with  $\nu = \min\{\alpha, 1 - 3\alpha\}$ .

The next proposition gathers the basic properties of the approximate solution  $u_\varepsilon$  that will be used in the proof of Theorem 3.1.1.

**Proposition 3.1.2.** *Fix  $\varepsilon > 0$ . If  $u_\varepsilon$  denotes the solution of (3.1.2), then*

- (P1)  $u_\varepsilon$  is unique and has a bound independent of  $\varepsilon$  (see [88]).  
 (P2)  $u_\varepsilon$  is continuous on  $U$ .  
 (P3) If  $g$  is  $C^{0,1}(\partial U)$ , then there exists  $\eta \in (0, 1)$  and  $C > 0$  depending on the data such that, for all  $x, y \in U$ , we have

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C \left( 1 + \|g\|_{C^{0,1}(\partial U)} \right) \rho_\varepsilon(x, y).$$

with  $\rho_\varepsilon(x, y) := \max\{\text{dist}(x, y), \varepsilon^{1-\alpha}\}$ .



## 3.2 Proof of the rate of convergence

A technical difficulty arises because viscosity solutions of fully nonlinear uniformly elliptic equations are not in general  $C^2$ . To solve this issue, we rely on the regularity theory, in particular the  $W^{2,\sigma}$  and  $W^{3,\sigma}$  estimates (here  $\sigma > 0$  is small (see [34] and [2, Lemma 5.2]) which give quadratic expansions for solutions of constant-coefficient equations in sets of large measure. This is essentially the same idea as the one used by Caffarelli and Souganidis in [36]. Moreover, we decompose the proof of Theorem 3.1.1 into two steps. First we study the rate of convergence of the approximation  $u_\varepsilon$  near the boundary  $\partial U$  which is assumed to be smooth (more precisely  $C^2$ ) by using the properties established in Proposition 3.1.2. Then we will examine the rate of convergence inside the domain, which is the core of the proof.

Now we state the elliptic  $W^{3,\sigma}$  estimate (see Appendix A for a direct proof). Define, for  $u : U \rightarrow \mathbb{R}$ , the quantity

$$\Psi(u, U)(x) := \inf \left\{ A \geq 0 : \exists (p, M) \in \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } \forall y \in U, \right. \\ \left. |u(y) - u(x) + p \cdot (x - y) + \frac{1}{2}(x - y) \cdot M(x - y)| \leq \frac{1}{6}A|x - y|^3 \right\}. \quad (3.2.1)$$

**Proposition 3.2.1** (Elliptic  $W^{3,\sigma}$  estimate). *Assume that  $\mathcal{F}$  is uniformly elliptic and  $u \in C(B_1)$  solves (3.1.1) in  $Q_1$ . Then there are universal constants  $C, \sigma > 0$  such that, for all  $\kappa > 0$ ,*

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{(1 + |\xi|) \sup_{B_1} |u| + |\mathcal{F}(0)|} \right)^{-\sigma}. \quad (3.2.2)$$

Then we state the other regularity tool we will use to derive the algebraic rate for the approximate scheme solution inside the domain. The proof of this proposition is postponed to Section 3.4.

**Proposition 3.2.2.** *Let  $U \subseteq \mathbb{R}^d$  be a  $C^2$ -bounded domain. Assume that  $u \in C(U)$  and  $v : U \rightarrow \mathbb{R}$  satisfy*

$$\mathcal{P}_{1,\Lambda}^-(D^2u) \leq 1 \quad \text{in } U \quad (3.2.3)$$

and

$$F_\varepsilon[v] \geq -1 \quad \text{in } U.$$

Fix  $0 < \delta \leq 1$  and define, for each  $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ , a function  $\Phi_{p,q} : U \times U \rightarrow \mathbb{R}$  by

$$\Phi_{p,q}(x, y) := u(x) - v(y) - \frac{1}{2\delta}|x - y|^2 - p \cdot x - q \cdot y.$$

Assume that  $V \subseteq \mathbb{R}^{2d}$  and  $W \subseteq \mathbb{R}^{2d}$  are such that  $\overline{V} \subseteq U \times U$  and, for each  $(p, q) \in W$ , there exists  $(x, y) \in V$  such that

$$\Phi_{p,q}(x, y) = \sup \{ \Phi_{p,q}(\xi, \eta) : (\xi, \eta) \in U \times U \}.$$

Then there exists a constant  $C > 1$  depending only on  $d, \Lambda$  and  $\text{diam } U$  such that for all  $\rho > 0$ ,

$$|W| \leq C\delta^{-2d} \left( 1 + \frac{\varepsilon^{1-\alpha}}{\rho} \right)^{2d} |V + B_\rho \times B_\rho|.$$

### 3.2.1 Rate near the boundary

Under the assumption of uniform ellipticity for the operator  $\mathcal{F}$ , we know that the viscosity solution of (3.1.1) satisfies  $u \in C^{0,\alpha}(\overline{U})$ . Moreover,  $u$  satisfies in the classical sense

$$u = g \quad \text{on } \partial U, \quad (3.2.4)$$

its proof being postponed to the appendix.

**Proposition 3.2.3** (Rate near the boundary). *Let  $U$  be a  $C^2$  bounded domain and  $U(r)$  be defined by (3.1.5). Then there exists a constant  $C$  depending only on  $\lambda, \Lambda, d$  and  $U$  such that, for all  $r > 0$ , we have*

$$\|u - u_\varepsilon\|_{L^\infty(U(r))} \leq 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1) C \max\{\varepsilon^{1-\alpha}, r\}.$$

*In particular, if  $g \in C^{0,1}(\partial U)$ , it follows that there exists a constant  $C$  such that*

$$\|u - u_\varepsilon\|_{L^\infty(U(r))} \leq C(\|g\|_{C^{0,1}(\partial U)} + 1) \max\{\varepsilon^{1-\alpha}, r\}.$$

*Proof.* Let  $x \in \overline{U}$  and  $x_0 \in \partial U$  such that  $\|x - x_0\| = \text{dist}(x, \partial U)$ . By Appendix 3.5, we know that  $u = g$  on  $\partial U$  in the classical sense. We get

$$\begin{aligned} |u(x) - u_\varepsilon(x)| &\leq |u(x) - u(x_0)| + |u(x_0) - u_\varepsilon(x)| \\ &\leq C\|x - x_0\| + |g(x_0) - u_\varepsilon(x)|. \end{aligned}$$

by using that  $u \in C^{0,\alpha}(\overline{U})$  in the last line. By Proposition 3.3.8, there exists a constant  $C > 0$  such that

$$|g(x_0) - u_\varepsilon(x)| \leq 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1)C \max\{\varepsilon^{1-\alpha}, r\}$$

and we get the desired estimate.  $\square$

### 3.2.2 Rate far the boundary

**Proposition 3.2.4.** *Let  $U$  be a  $C^2$ -bounded domain. Assume that  $\mathcal{F}$  and  $F_\varepsilon$  satisfy respectively (F1)-(F2) and (S1)-(S2). Assume that the functions  $u, u_\varepsilon \in C(\overline{U})$  are respectively solutions of (3.1.1) and (3.1.11). Then there exists an exponent  $0 < \eta < 1$  and a constant  $C$  depending only on the ellipticity constants  $\lambda, \Lambda$ , the dimension  $d$  and  $\|g\|_{C^{0,1}}$  such that for  $r \geq 5\varepsilon^{1-\alpha}$ , we have*

$$\sup_{x \in U \setminus U(r)} (u_\varepsilon - v) \leq C\varepsilon^\eta.$$

*Proof.* Throughout the proof,  $C$  and  $c$  denote positive constants which depend on  $d, \Lambda$  and  $U$  but may be different in each instance. Arguing by contradiction, we assume that there exists  $\varepsilon > 0$  small enough such that  $E \geq C\varepsilon^q$  for a certain  $q > 0$ .

*Step 1.* We make several initial observations. First we may assume without loss of generality that  $U \subseteq B_1$  and  $K_1 = 1$ , using some rescaling/normalizing arguments. Second, by comparing  $v$  to the function  $x \mapsto u(x) + \frac{1}{2}E^\eta(1 - |x|^2)$  with  $\eta > 0$  to be selected below, we may replace the equation for  $u$  by

$$\xi u + \mathcal{F}(D^2 u) = cE^\eta \quad \text{in } B_1. \quad (3.2.5)$$

Indeed, otherwise, we may replace  $E$  by  $\frac{1}{2}E$  and  $u$  by the solution of the Dirichlet problem for (3.2.5) with the same boundary condition. Third, in view of the bound  $K_1 \leq 1$  and the smoothness of  $U$ , the global Hölder estimates yield, for  $\sigma(d, \Lambda) \in (0, 1)$ ,

$$\|u\|_{C^\sigma(\overline{U})} + \|u_\varepsilon\|_{C^\sigma(U)} \leq C. \quad (3.2.6)$$

(attention à la régularité de la solution discrète : surement enlever la dépendance en  $u_\varepsilon$ ).

Since  $u = g$ , the triangle inequality gives, for every  $x, y \in U$ ,

$$|u(x) - u_\varepsilon(y)| \leq C \text{dist}(\{x, y\}, \partial U)^\sigma + C|x - y|^\sigma + C(1 + \|g\|_{C^{0,1}(\partial U)} \varepsilon^{1-\alpha}) \quad (3.2.7)$$

We may assume without loss of generality that

$$C(1 + \|g\|_{C^{0,1}(\partial U)} \varepsilon^{1-\alpha}) \leq \frac{1}{8}E \quad (3.2.8)$$

since otherwise we are done. For convenience we may take  $0 < \sigma \leq 1/2$ .

**Step 2.** We consider the auxiliary function  $\Phi : U \times U \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\Phi_{p,q}(x, y) := u(x) - u_\varepsilon(y) - \frac{1}{2\delta} (|x - y|^2) - p \cdot x - q \cdot y,$$

for some  $\delta > 0$  to be determined. Choose  $x_0 \in U$  such that  $\Phi(x_0, x_0, 0, 0) = E$ . Set  $r := \min\{\frac{1}{8}E, 1\}$ . Given  $p, q \in B_r$  and  $\max\{|\alpha|, |\beta|\} \leq r$ , we compute

$$\Phi_{p,q}(x_0, x_0) := E - (|p| + |q|)|x_0| \geq \frac{3}{4}E,$$

and estimate

$$\begin{aligned} \Phi_{p,q}(x, y) &= u(x) - v(y) - \frac{1}{2\delta} (|x - y|^2) - p \cdot x - q \cdot y \\ &\leq C \operatorname{dist}(\{x, y\}, \partial U)^\sigma + C|x - y|^\alpha - \frac{1}{2\delta}|x - y|^2 + 2r + C(1 + \|g\|_{C^{0,1}(\partial U)} \varepsilon^{1-\alpha}) \\ &\leq C \operatorname{dist}(\{x, y\}, \partial U)^\sigma + \frac{1}{4}E + \left(CE^{-(2-\sigma)/\sigma} - \frac{1}{2\delta}\right)|x - y|^2 + 2r + C(1 + \|g\|_{C^{0,1}(\partial U)} \varepsilon^{1-\alpha}) \\ &\leq C \operatorname{dist}(\{x, y\}, \partial U)^\sigma + \frac{5}{8}E \end{aligned}$$

where in the third line we used Young's inequality and to get the fourth line of the inequality, we use (3.2.8) and we must impose the condition  $\delta \leq cE^{(2-\sigma)/\sigma}$ . Then we may fix

$$\delta = cE^{(2-\sigma)/\sigma} \quad (3.2.9)$$

so that, for all  $(p, q) \in B_r$ , the map

$$(x, y) \mapsto \Phi_{p,q}(x, y)$$

attains its supremum in  $U \times U$  on  $\overline{U}_s \times \overline{U}_s$  where  $s := cE^{1/\sigma}$ . Here we have denoted

$$U_s := \{x \in U : \operatorname{dist}(x, \partial U) > s\}.$$

Let  $Z$  be the set of points where such supremums are attained:

$$Z := \left\{ (x, y) \in U_s \times U_s : \exists (p, q) \in B_r \times B_r \text{ s.t. } \Phi_{p,q}(x, y) = \sup_{U \times U} \Phi_{p,q} \right\}$$

We apply Proposition 3.2.2 to get

$$|Z| \geq \delta^{2d} r^{2d} \geq c(E^{(2-\sigma)/\sigma})^{2d} E^{2d} = cE^{4d/\sigma}.$$

Let  $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the projection onto the first  $d$  variables, i.e.,  $\pi_1(x, z) := x$ . Observe that

$$|\pi_1(\tilde{Z}_\rho)| \geq |U_s|^{-1}|Z|. \quad (3.2.10)$$

Finally, we show that for every  $(x, y) \in Z$ , we can see from  $\Phi_{p,q}(x, y)$  for some  $p, q \in B_1$  and  $\sigma < 1/2$  that

$$|x - y|^2 \leq C\delta \leq CE^{(2-\sigma)/\sigma} \leq CE^3, \quad (3.2.11)$$

**Step 3.** We show that there are points  $(x, t, y, s) \in Z$  such that  $v$  has an appropriate quadratic expansion at  $(y, s)$ . Let  $P_\kappa$  be the set of points at which  $u$  has a quadratic expansion with both a quadratic term of size  $\kappa > 0$  and a cubic error term of size  $\kappa > 0$ :

$$\begin{aligned} P_\kappa &:= \{x \in U : \exists (A, \xi) \in \mathbb{R} \times \mathcal{S}_d \times \mathbb{R}^d \text{ s.t. } |A| \leq \kappa \text{ and, for all } \forall y \in U, \\ &\quad |v(y) - v(x) + \xi \cdot (y - x) + \frac{1}{2}(x - y) \cdot A(x - y)| \leq \frac{1}{6}\kappa|x - y|^3\}. \end{aligned}$$

According to the elliptic  $W^{3,\sigma}$  estimate (see Proposition 3.2.1), we have

$$|U \setminus P_\kappa| \leq C\kappa^{-\sigma},$$

where the exponent  $\sigma > 0$  depends only on  $d, \lambda$  and  $\Lambda$  (we may reuse the symbol  $\sigma$  by taking the minimum of this  $\sigma$  with the one from Step 1). In view of (3.2.10), we have, for every  $\kappa \geq CE^{-4d/\sigma^2}$ ,

$$|U \setminus P_\kappa| < |\pi_1(Z)|C\kappa^{-\sigma},$$

Therefore, we take

$$\kappa = CE^{-4d/\sigma^2} \quad (3.2.12)$$

to be a fixed constant, which will be selected below. In particular,  $\pi_1(Z) \cap P_\kappa \neq \emptyset$ .

*Step 4.* By the previous step, there exists  $(x_1, y_1) \in Z$  with  $y_1 \in P_\kappa$ . Select  $p, q \in B_r$  and such that

$$\Phi_{p,q}(x_1, y_1) = \sup_{x, y \in U \times U} \Phi_{p,q}(x, y) \quad (3.2.13)$$

and  $(A, \xi) \in \mathcal{S}_d \times \mathbb{R}^d$  such that  $|A| \leq \kappa$  and, for all  $(y, s) \in U$

$$|u(y) - u(y_1) + \xi \cdot (y - y_1) + \frac{1}{2}(y - y_1) \cdot A(y - y_1)| \leq \frac{1}{6}\kappa|y - y_1|^3. \quad (3.2.14)$$

Note that  $\xi u(y_1) + \mathcal{F}(A) = E^\eta$ , since  $u$  satisfies  $\xi u + \mathcal{F}(D^2 u) = E^\eta$  in  $U$  and  $u$  is touched from above and below at  $y_1$  by cubic polynomials with Hessians equal to  $A$  at  $y_1$ . Combining (4.2.13) and (4.2.14) gives

$$u_\varepsilon(x_1) - \phi(y_1) - \frac{1}{2\delta}|x_1 - y_1|^2 - p \cdot x_1 = \sup_{x, y \in \overline{U}_s \times \overline{U}_s} \left( u_\varepsilon(x) - \phi(y) - \frac{1}{2\delta}|x - y|^2 - p \cdot x \right) \quad (3.2.15)$$

where  $\phi$  is the cubic polynomial defined by

$$\phi(y) = u(y_1) + (\xi + q) \cdot (y - y_1) + \frac{1}{2}(y - y_1) \cdot A(y - y_1) + \frac{1}{6}\kappa(|y - y_1|^3).$$

Observe that, for each  $x \in U$ ,

$$\begin{aligned} \inf_{y \in U} \left( \phi(y) + \frac{1}{2\delta}|x - y|^2 \right) &\leq \phi(y_1 + x - x_1) - \frac{1}{2\delta}|x_1 - y_1|^2 \\ &\leq u(y_1) + (\xi + q) \cdot (x - x_1) + \frac{1}{2}(x - x_1) \cdot A(x - x_1) + \frac{1}{6}\kappa|x - x_1|^3 + \frac{1}{2\delta}|x_1 - y_1|^2. \end{aligned}$$

Inserting this into (4.2.16), using  $u(y_1) = \varphi(y_1)$  and after rearranging the terms, we obtain

$$u_\varepsilon(x_1) = \sup_{x \in U} \left( u_\varepsilon(x) - (\xi + q + p) \cdot (x - x_1) - \frac{1}{2}(x - x_1) \cdot A(x - x_1) - \frac{1}{6}\kappa|x - x_1|^3 \right)$$

Observe this sup can be restrained to the set  $B(x_1, 2\varepsilon^{1-\alpha})$ . Hence, the inequality  $|x - x_1| \leq 2\varepsilon^{1-\alpha}$  holds true. By taking into account these two remarks, we define

$$\psi(x) := u(x_1) + (\xi + q + p) \cdot (x - x_1) + \frac{1}{2}(x - x_1) \cdot A(x - x_1) + \frac{1}{6}\varepsilon^{1-\alpha}\kappa|x - x_1|^3.$$

Then the point  $x_1$  is a maximum point of  $u_\varepsilon - \psi$  on  $B(x_1, 2\varepsilon^{1-\alpha})$ . By monotonicity of  $S_\varepsilon$ , we get

$$F_\varepsilon[\psi](x_1) + \frac{1 - e^{-\xi\varepsilon^2}}{\varepsilon^2}(u_\varepsilon - \psi)(x_1) \leq F_\varepsilon[u_\varepsilon](x_1). \quad (3.2.16)$$

We evaluate the r.h.s. above by using (S2) which yields

$$\left| F_\varepsilon[\psi](x_1) - (\xi u(x_1) + \mathcal{F}\left(A + \frac{1}{3}\kappa\varepsilon^{1-\alpha}\right)) \right| \leq K_1\varepsilon^\nu.$$

by using  $\psi(x) = u(x_1)$  and  $D^2\psi(x_1) = A + \frac{1}{3}\kappa\varepsilon^{1-\alpha}$ . Therefore, we get by (F1) that

$$\xi u(x_1) + \mathcal{F}(A) - \frac{\Lambda d}{3}\kappa\varepsilon^{1-\alpha} - K_1\varepsilon^\nu + \frac{1 - e^{-\xi\varepsilon^2}}{\varepsilon^2}(u_\varepsilon - u)(x_1) \leq 0$$

By combining  $\xi u(y_1) + \mathcal{F}(A) = E^\eta$ , the computation

$$\xi u(x_1) \geq \xi u(y_1) - \xi|u(y_1) - u(x_1)| \geq \xi u(y_1) - C|x_1 - y_1|^\alpha \geq \xi u(y_1) - C\delta^{\alpha/2},$$

by using that  $|x_1 - y_1|^2 \leq \delta$  by Step 2 and using that  $(u_\varepsilon - u)(x_1) \geq 0$ , we get that

$$E^\eta \leq K_1\varepsilon^\nu + \frac{\Lambda d}{3}\kappa\varepsilon^{1-\alpha} + C\delta^{\alpha/2}.$$

By inserting the expressions of  $\delta$  and  $\kappa$  given by (3.2.9) and (3.2.12) and using that  $\varepsilon \leq CE^{1/q}$ , we get

$$E^\eta \leq CK_1E^{\nu/q} + \frac{\Lambda d}{3}E^{-4d/\sigma^2 + (1-\alpha)/q} + CE^{(2-\sigma)\alpha/(2\sigma)}.$$

By selecting  $\eta > 0$  such that

$$\eta < \frac{(2-\sigma)\alpha}{2\sigma}$$

and using that  $\nu \leq \alpha < 1/3 < 1 - \alpha$ , the last expression reduces to:

$$E^\eta < cE^{(1-\alpha)/q - 4d/\sigma^2}$$

which yields a contradiction by the choice for  $q$  given by

$$q > \frac{1-\alpha}{\eta + 4d/\sigma^2}. \quad \square$$

### 3.3 The dynamic programming schemes : presentation and properties

#### 3.3.1 Presentation of the games

The games are presented in [88, Section 2.4] in the general setting. We refer to the introduction of the thesis or Chapter 2 for more details concerning the general approach. For sake of completeness, we give here an exposition in our framework corresponding to a uniform operator depending only on the Hessian.

The game considered here is a variant to the game corresponding to the Neumann boundary condition presented in Chapter 1. There are still a discount factor and play stops when the score of Helen gets too large in absolute value. The main difference is the fact that play stops when Helen gets to  $\partial U$ . The boundary condition  $g$  is assumed to be a bounded continuous function on  $\partial U$ . It enters the game as an “exit bonus”; since the final position can (slightly) be outside  $U$ , we shall assume that  $g$  has been extended to a continuous function defined on a neighborhood of  $U$ . Beside the parameters  $\alpha, \beta, \gamma$  already introduced, two new parameters  $m$  and  $M$  are needed. Both are positive constants;  $M$  serves to cap the score  $m$  determines what happens when the cap is reached. In due course, we can choose them such that  $m = M - 1$  and require that  $M$  be sufficiently large (see [88, Section 2.4] for more details) Like the choices of  $\alpha, \beta, \gamma$ , the parameters  $M$  and  $m$  are used to define the game but do not influence the resulting PDE.

We proceed in two steps:

- First we introduce  $U^\varepsilon(x, z)$ , the optimal worst-case present value of Helen’s wealth if the initial stock price is  $x$  and her initial wealth is  $-z$ .

- Then we define  $u_\varepsilon(x)$  and  $v_\varepsilon(x)$  as the maximal and minimal initial debt Helen should have at the initial time to break even upon exit.

The definition of  $u_\varepsilon(x, z)$  for  $x \in \overline{U}$  involves a game similar to that of the last section:

1. Initially, at time  $t_0 = 0$ , the stock price is  $x_0 = x$  and Helen's debt is  $z_0 = z$ .
2. Suppose, at time  $t_j = j\varepsilon^2$ , the stock price is  $x_j$  and Helen's debt is  $z_j$  with  $|z_j| < M$ . Then Helen chooses a vector  $p_j \in \mathbb{R}^d$  and a matrix  $\Gamma_j \in \mathcal{S}_d$ , restricted in magnitude by (3.1.9). Knowing these choices, Mark determines the next stock price  $x_{j+1} = x_j + \Delta x$  so as to degrade Helen's outcome. The increment  $\Delta x$  allows to model the reflection exactly as in the previous subsections. Mark chooses the next position  $x_{j+1} = x_j + \Delta x_j \in \mathbb{R}^d$  such that  $\Delta x_j$  is restricted in magnitude by (3.1.10). Helen experiences a loss at time  $t_j$  of

$$\delta_j = p_j \cdot \Delta x_j + \frac{1}{2} \langle \Gamma_j \Delta x_j, \Delta x_j \rangle + \varepsilon^2 \mathcal{F}(\Gamma_j). \quad (3.3.1)$$

As a consequence, her time  $t_{j+1} = t_j + \varepsilon^2$  debt becomes

$$z_{j+1} = e^{\lambda \varepsilon^2} (z_j + \delta_j),$$

where the factor  $e^{\lambda \varepsilon^2}$  takes into account her interest payments.

3. If  $|z_{j+1}| < M$  and  $x_{j+1} \notin U$ , then the game terminates, and Helen gets an “exit payoff” worth  $g(x_{j+1})$  at time  $t_{j+1}$ . If  $z_{j+1} \geq M$ , then the game terminates, and Helen pays a “termination-by-large-debt penalty” worth  $e^{\lambda \varepsilon^2} (m - \delta_j)$  at time  $t_{j+1}$ . Similarly, if  $z_{j+1} \leq -M$ , then the game terminates, and Helen receives a “termination-by-large-wealth bonus” worth  $e^{\lambda \varepsilon^2} (m + \delta_j)$  at time  $t_{j+1}$ . If the game stops this way, we call  $t_{j+1}$  the “ending index”  $t_K$ .
4. If the game has not terminated then Helen and Mark repeat this procedure at time  $t_{j+1} = t_j + \varepsilon^2$ . If the game never stops, the “ending index”  $t_K$  is  $+\infty$ .

Since the game is stationary (nothing distinguishes time 0), the associated dynamic programming principle is that for  $|z| < M$ ,

$$U^\varepsilon(x, z) = \sup_{p, \Gamma} \inf_{\Delta \hat{x}} \begin{cases} e^{-\lambda \varepsilon^2} U^\varepsilon(x', z'), & \text{if } |z'| < M, \\ -z - m, & \text{if } z' \geq M, \\ -z + m, & \text{if } z' \leq -M, \end{cases} \quad (3.3.2)$$

where  $x' = x + \Delta x$  and  $z' = e^{\lambda \varepsilon^2} (z + \delta)$ , with  $\delta$  defined as in (3.3.1). Here  $p$ ,  $\Gamma$  and  $\Delta x$  are constrained as usual by (3.1.9)–(3.1.10), and we write sup / inf rather than max / min since it is no longer clear that the optima are achieved (since the right-hand side is now a discontinuous function of  $p$ ,  $\Gamma$  and  $\Delta \hat{x}$ ). The definitions of  $u_\varepsilon$  and  $v_\varepsilon$  are the following. For  $x_0 \in U$ , set

$$u_\varepsilon(x_0) := \sup\{z_0 : U^\varepsilon(x_0, z_0) > 0\}, \quad (3.3.3)$$

$$v_\varepsilon(x_0) := \inf\{z_0 : U^\varepsilon(x_0, z_0) < 0\}. \quad (3.3.4)$$

Since  $\mathcal{F}$  is non-decreasing in  $z$ , [88, Lemma 5.4] implies the function  $U^\varepsilon$  is strictly decreasing in  $z$  and  $v^\varepsilon = u_\varepsilon$ . In that case, the dynamic programming principle given by [88, Proposition 2.5] reduces to the dynamic programming equation given by the following proposition.

**Proposition 3.3.1.** *Assume that  $m, M$  are chosen sufficiently large such that  $\|u\|_{L^\infty(U)} \leq m \leq M - 1$ . Then, for all  $\varepsilon$  small enough, any  $x \in U$  satisfies*

$$u_\varepsilon(x) = S_\varepsilon[x, u_\varepsilon].$$

The operator  $S_\varepsilon$  defined by (3.1.8) clearly satisfies the three following properties:

- For all  $\phi \in C(\overline{U})$ ,  $S_0[x, \phi] = \phi(x)$ .

- $S_\varepsilon$  is monotone, i.e. if  $\phi_1 \leq \phi_2$ , then  $S_\varepsilon[x, \phi_1] \leq S_\varepsilon[x, \phi_2]$ .
- For all  $\phi \in C(\overline{U})$  and  $c \in \mathbb{R}$ ,  $S_\varepsilon[x, c + \phi] = e^{-\xi\varepsilon^2}c + S_\varepsilon[x, \phi]$ .

We next recall the statement of the convergence established by Kohn and Serfaty [88] in our particular case.

**Proposition 3.3.2** (Kohn and Serfaty [88]). *Consider the stationary boundary-value problem (3.1.1) where  $\mathcal{F}$  is elliptic and satisfies some regularity assumptions, and  $g$  is a uniformly bounded continuous function. Assume the parameters of the game  $\alpha, \beta, \gamma$  satisfy (3.1.6)–(3.1.7),  $M$  is sufficiently large and  $m = M - 1$ . Then  $u_\varepsilon$  is well-defined when  $\varepsilon$  is sufficiently small, and they satisfy  $u_\varepsilon \leq m$ . Its semi-relaxed limits  $\overline{u}$  and  $\underline{u}$  coincide and are a viscosity solution of (3.1.1). It follows that  $u_\varepsilon$  converge locally uniformly in  $U$  to the unique viscosity solution of (3.1.1).*

### 3.3.2 Proof of the assumptions (S1) and (S2) on $F_\varepsilon$

Before giving the proof of the properties (S1) and (S2) satisfied by  $F_\varepsilon$ , we start by a preliminary lemma, which is the key tool for the comparison principle associated to the game. Then we will give a global comparison principle on the scheme which implies the definition of some global sub and super solutions associated to the scheme.

**Lemma 3.3.3** (Local comparison principle). *For all  $x \in U$  and all  $H \supseteq B(x, \varepsilon^{1-\alpha})$ , we have*

$$\frac{1}{\varepsilon^2} \left( u(x) - v(x) - e^{-\xi\varepsilon^2} \sup_{U \cap H} (u - v) \right) \leq F_\varepsilon[u](x) - F_\varepsilon[v](x).$$

*Proof.* To prove it, it suffices to write

$$u(x) \leq v(x) + \sup_{U \cap H} (u - v)$$

and we get by monotonicity of the operator  $S_\varepsilon$  that

$$\begin{aligned} S_\varepsilon[x, u] - u(x) &\leq S_\varepsilon[x, v] - u(x) + e^{-\xi\varepsilon^2} \sup_{U \cap H} (u - v) \\ &\leq (S_\varepsilon[x, v] - v(x)) + v(x) - u(x) + e^{-\xi\varepsilon^2} \sup_{U \cap H} (u - v). \end{aligned}$$

By dividing by  $\varepsilon^2$ , it follows the desired inequality.  $\square$

This local comparison principle gives the comparison principle (S1) but also has a global version.

(S1G) **(Global comparison principle)** Assume that  $u$  and  $v$  satisfy

$$F_\varepsilon[u] \leq F_\varepsilon[v] \quad \text{on } U$$

Then

$$\sup_U (u - v) \leq 0.$$

In particular, it is clear by this last property that the solution  $u_\varepsilon$  given by the scheme is unique. Moreover, the local comparison principle given by Lemma 3.3.3 authorizes us to define the notion of approximate sub solution and super solution for the scheme. We say that a real-valued bounded function  $u(x)$  defined on  $U$  is a  $\varepsilon$ -discrete supersolution of the stationary problem (3.1.11) if

$$F_\varepsilon[u] \geq 0 \quad \text{in } U. \tag{3.3.5}$$

Similarly, a real-valued bounded function  $u(x)$  defined on  $U$  is a  $\varepsilon$ -discrete subsolution of the stationary problem (3.1.11) if

$$F_\varepsilon[u] \leq 0 \quad \text{in } U. \tag{3.3.6}$$

A discrete solution of (3.1.11) is a continuous function  $u$  that is both a  $\varepsilon$ -discrete subsolution and a  $\varepsilon$ -discrete supersolution of (3.1.11).

We are now ready to prove the assertions (S1), (S1G) and (S2) on the elliptic schemes.

*Proof of (S1), (S1G) and (S2).* By applying Lemma 3.3.3, we deduce that

$$\sup_H \{u(x) - v(x)\} \leq e^{-\xi\varepsilon^2} \sup_{H \cup \partial_\varepsilon H} \{u(x) - v(x)\}.$$

Since  $0 < e^{-\xi\varepsilon^2} < 1$ , the sup associated to the r.h.s. reduces to the sup on  $\partial_\varepsilon H$ .

Since  $u$  and  $v$  are bounded, the sup is well-defined. Let us take a sequence  $(x_k)_k$  of points of  $U$  such that

$$(u - v)(x_k) \xrightarrow{k \rightarrow +\infty} \sup_U (u - v).$$

By applying Lemma 3.3.3, we get

$$\frac{1}{\varepsilon^2} \left( u(x_k) - v(x_k) - e^{-\xi\varepsilon^2} \sup_U (u - v) \right) \leq F_\varepsilon[u](x_k) - F_\varepsilon[v](x_k). \quad (3.3.7)$$

Moreover, the right-hand side of (3.3.7) is nonpositive for all integer  $k$  since  $u$  and  $v$  are respectively  $\varepsilon$ -discrete sub and supersolution. Thus, by taking the limit  $k \rightarrow +\infty$ , one obtains

$$\frac{1}{\varepsilon^2} \left( 1 - e^{-\xi\varepsilon^2} \right) \sup_U (u - v) \leq 0.$$

Using that  $1 - e^{-\xi\varepsilon^2} > 0$ , we conclude that  $\sup_U (u - v) \leq 0$ .

The consistency estimate (S2) is a corollary of the consistency analysis led in [88]. By following carefully the proof given [88], we have

$$\begin{aligned} -M\varepsilon^{3-3\alpha} &\leq S_\varepsilon[x, z, \phi] - \phi(x) + \varepsilon^2 [\mathcal{F}(D^2\phi(x)) + \xi\phi(x)] \\ &\leq \varepsilon^2 \sup_{\substack{\|p - D\phi(x)\| \leq \varepsilon^\mu \\ \|\Gamma - D^2\phi(x)\| \leq \varepsilon^\alpha}} |\mathcal{F}(\Gamma) - \mathcal{F}(D^2\phi(x))| + M\varepsilon^{3-3\alpha}, \end{aligned}$$

where  $M$  is defined by  $M = \sup\{\|D^3\phi(y)\| : y \in B(x, \varepsilon^{1-\alpha})\}$ . Then the consistency estimate (S2) follows by using the uniform ellipticity of the operator  $\mathcal{F}$ .  $\square$

### 3.3.3 Proof of Proposition 3.1.2

In this subsection, we are going to work for a certain  $\varepsilon > 0$  and study the properties of the solution  $u^\varepsilon$  of the dynamic programming equation. The uniqueness as the boundedness of the approximate solution comes directly from the fixed-point argument contained in [88, Section 5]. The continuity on  $U$  is proved in Section 3.3.3.1. It remains to prove the Lipschitz estimate given by (P3).

#### 3.3.3.1 Continuity of the score functions

At first sight, we do not know anything about the regularity of the approximate solution  $u^\varepsilon$ . To get the continuity property, we are going to use the stability of the elliptic  $\varepsilon$ -sub and supersolutions by passing to relaxation. Denote the upper semicontinuous envelope  $u^*$  of a bounded function  $u : U \rightarrow \mathbb{R}$  by

$$u^*(x) := \limsup_{y \rightarrow x} u(y), \quad x \in U.$$

the limit being considered by taking  $y \in U$ . The lower semicontinuous envelope of  $u$  is  $u_* := -(-u)^*$ . The function  $u^*$  is upper semicontinuous,  $u_*$  is lower semicontinuous, and  $u_* \leq u \leq u^*$ .



**Lemma 3.3.4** (Stability of discrete sub- and supersolution by relaxation). *We consider the semi-discrete scheme defined by (3.1.11). Then,*

- *If  $w$  is a  $\varepsilon$ -discrete subsolution of (3.1.11), then  $w^\star$  is also a  $\varepsilon$ -discrete subsolution of (3.1.11).*
- *If  $v$  is a  $\varepsilon$ -discrete supersolution of (3.1.11), then  $v_\star$  is also a  $\varepsilon$ -discrete supersolution of (3.1.11).*

We can now give the proof of the continuity of the discrete solutions  $u_\varepsilon$ .

*Proof of Proposition 3.1.2 item (P1).* Denote the upper and lower semi-relaxed limits  $(u_\varepsilon)^\star$  and  $(u_\varepsilon)_\star$  of  $u_\varepsilon$ . By Lemma 3.3.4, we get that

$$F_\varepsilon[(u_\varepsilon)^\star] \leq 0 \quad \text{and} \quad F_\varepsilon[(u_\varepsilon)_\star] \geq 0 \quad \text{on } U.$$

By (S1G), we deduce that

$$\max_U((u_\varepsilon)^\star - u_\varepsilon) \leq 0 \quad \text{and} \quad \max_U(u_\varepsilon - (u_\varepsilon)_\star) \leq 0. \quad (3.3.8)$$

In addition, by the definition of the semi-relaxed limits, it is clear that  $(u_\varepsilon)_\star \leq u_\varepsilon \leq (u_\varepsilon)^\star$ . By using (3.3.8), we conclude that  $u = (u_\varepsilon)^\star = (u_\varepsilon)_\star$  on  $U$ . In particular,  $u_\varepsilon$  is continuous on  $U$ .  $\square$

It remains to prove Lemma 3.3.4.

*Proof of Lemma 3.3.4.* Let us examine the first assertion about the discrete subsolutions. For sake of notational simplicity, up to change  $u$  into  $\hat{u}$ , we fix  $x \in U$  such that  $\text{dist}(x, \partial U) > \varepsilon^{1-\alpha}$ . Let  $\{x_k\}_{k=1}^\infty \subseteq U$  be a sequence converging to  $x$  as  $k \rightarrow \infty$  and for which

$$w(x_k) \rightarrow w^\star(x).$$

We claim that

$$\varepsilon^2 F_\varepsilon[w^\star](x) = w^\star(x) - S_\varepsilon[x, w^\star] \leq 0. \quad (3.3.9)$$

Since  $w$  is a  $\varepsilon$ -discrete subsolution, we know that, for all  $k \geq 0$ , we have

$$w(x_k) \leq S_\varepsilon[x_k, w].$$

By taking the limit  $k \rightarrow +\infty$ , we obtain

$$w^\star(x) \leq \limsup_{k \rightarrow +\infty} S_\varepsilon[x_k, w]. \quad (3.3.10)$$

We now give an upper bound on the right-hand side of (3.3.10). For all  $k \in \mathbb{N}$ , there exist  $p_k, \Gamma_k$  constrained by (3.1.9) such that

$$S_\varepsilon[x_k, w] \leq \min_{\Delta x} \left[ e^{-\xi \varepsilon^2} w(x_k + \Delta x) - \left( p_k \cdot \Delta x + \frac{1}{2} \langle \Gamma_k \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma_k) \right) \right] + \delta. \quad (3.3.11)$$

By compactness, there exists  $\|\tilde{p}\| \leq \varepsilon^{-\beta}$ ,  $\|\tilde{\Gamma}\| \leq \varepsilon^{-\gamma}$  such that

$$p_k \rightarrow \tilde{p}, \quad \Gamma_k \rightarrow \tilde{\Gamma} \quad \text{as } k \rightarrow +\infty. \quad (3.3.12)$$

By considering the particular choice  $p = \tilde{p}$  and  $\Gamma = \tilde{\Gamma}$  in the definition of the max for  $S_\varepsilon$ , we get

$$S_\varepsilon[x, w^\star] \geq \min_{\Delta x} \left[ e^{-\xi \varepsilon^2} w^\star(x + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2} \langle \tilde{\Gamma} \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) \right].$$

Fix  $\tilde{\delta} > 0$ , we select  $\|\tilde{\Delta}x\| \leq \varepsilon^{1-\alpha}$  such that

$$\begin{aligned} e^{-\xi\varepsilon^2}w^*(x + \tilde{\Delta}x) - \left( \tilde{p} \cdot \tilde{\Delta}x + \frac{1}{2}\langle \tilde{\Gamma}\tilde{\Delta}x, \tilde{\Delta}x \rangle + \varepsilon^2\mathcal{F}(\tilde{\Gamma}) \right) \\ \leq \min_{\Delta x} \left( e^{-\xi\varepsilon^2}w^*(x + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2}\langle \tilde{\Gamma}\Delta x, \Delta x \rangle + \varepsilon^2\mathcal{F}(\tilde{\Gamma}) \right) \right) + \tilde{\delta}. \end{aligned}$$

For this choice for  $\tilde{\Delta}x$ , we obtain that

$$S_\varepsilon[x, w^*] \geq \left( e^{-\xi\varepsilon^2}w^*(x + \tilde{\Delta}x) - \left( \tilde{p} \cdot \tilde{\Delta}x + \frac{1}{2}\langle \tilde{\Gamma}\tilde{\Delta}x, \tilde{\Delta}x \rangle + \varepsilon^2\mathcal{F}(\tilde{\Gamma}) \right) \right) - \tilde{\delta}.$$

By using that  $w^* \geq w$  and  $x_k + \tilde{\Delta}x \rightarrow x + \tilde{\Delta}x$  we get

$$S_\varepsilon[x, w^*] \geq \limsup_{k \rightarrow +\infty} \left( e^{-\xi\varepsilon^2}w(x_k + \tilde{\Delta}x) - \left( \tilde{p} \cdot \tilde{\Delta}x + \frac{1}{2}\langle \tilde{\Gamma}\tilde{\Delta}x, \tilde{\Delta}x \rangle + \varepsilon^2\mathcal{F}(\tilde{\Gamma}) \right) \right) - \tilde{\delta}. \quad (3.3.13)$$

For all move  $\Delta x$ , we can write the decomposition

$$\begin{aligned} e^{-\xi\varepsilon^2}w(x_k + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2}\langle \tilde{\Gamma}\Delta x, \Delta x \rangle + \varepsilon^2\mathcal{F}(\tilde{\Gamma}) \right) \\ = e^{-\xi\varepsilon^2}w(x_k + \Delta x) - \left( p_k \cdot \Delta x + \frac{1}{2}\langle \Gamma_k \Delta x, \Delta x \rangle + \varepsilon^2\mathcal{F}(\Gamma_k) \right) \\ - \left( (\tilde{p} - p_k) \cdot \Delta x + \frac{1}{2}\langle (\tilde{\Gamma} - \Gamma_k)\Delta x, \Delta x \rangle + \varepsilon^2(\mathcal{F}(\tilde{\Gamma}) - \mathcal{F}(\Gamma_k)) \right). \quad (3.3.14) \end{aligned}$$

Since  $\mathcal{F}$  is Lipschitz, we have

$$\begin{aligned} \left| (\tilde{p} - p_k) \cdot \Delta x + \frac{1}{2}\langle (\tilde{\Gamma} - \Gamma_k)\Delta x, \Delta x \rangle + \varepsilon^2(\mathcal{F}(\tilde{\Gamma}) - \mathcal{F}(\Gamma_k)) \right| \\ \leq \varepsilon^{1-\alpha}\|\tilde{p} - p_k\| + (d\Lambda\varepsilon^2 + \varepsilon^{2-2\alpha})\|\tilde{\Gamma} - \Gamma_k\|. \quad (3.3.15) \end{aligned}$$

By combining (3.3.14) and (3.3.15) in the right-hand side of (3.3.13), we obtain

$$\begin{aligned} S_\varepsilon[x, w^*] \geq \limsup_{k \rightarrow +\infty} \left( e^{-\xi\varepsilon^2}w(x_k + \tilde{\Delta}x) - \left( p_k \cdot \tilde{\Delta}x + \frac{1}{2}\langle \Gamma_k \tilde{\Delta}x, \tilde{\Delta}x \rangle + \varepsilon^2\mathcal{F}(\Gamma_k) \right) \right. \\ \left. - \left\{ \varepsilon^{1-\alpha}\|\tilde{p} - p_k\| + (d\Lambda\varepsilon^2 + \varepsilon^{2-2\alpha})\|\tilde{\Gamma} - \Gamma_k\| \right\} - \tilde{\delta} \right). \end{aligned}$$

Then by passing to the min and using (3.3.11), we get

$$S_\varepsilon[w^*](x) \geq \limsup_{k \rightarrow +\infty} \left( S_\varepsilon[x_k, w] - \varepsilon^{1-\alpha}\|\tilde{p} - p_k\| - (d\Lambda\varepsilon^2 + \varepsilon^{2-2\alpha})\|\tilde{\Gamma} - \Gamma_k\| \right) - \delta - \tilde{\delta}.$$

By using (3.3.12), we get

$$S_\varepsilon[x, w^*] \geq \limsup_{k \rightarrow +\infty} S_\varepsilon[x_k, w] - (\delta + \tilde{\delta}),$$

Combining this inequality with (3.3.10) and sending  $\delta, \tilde{\delta} \rightarrow 0$ , we deduce that the desired estimate given by (3.3.9) holds.

We now give the proof of the second assertion. Fix  $x \in U$  and let  $\{x_k\}_{k=1}^\infty \subseteq U$  be a sequence converging to  $x$  as  $k \rightarrow \infty$  and for which  $v(x_k) \rightarrow v_\star(x)$ . We claim that

$$\varepsilon^2 F_\varepsilon[v_\star](x) = v_\star(x) - S_\varepsilon[x, v_\star] \geq 0. \quad (3.3.16)$$

Since  $v$  is a  $\varepsilon$ -discrete supersolution, we know that, for all integer  $k$ , we have

$$v(x_k) \geq S_\varepsilon[x_k, v].$$

By taking the limit  $k \rightarrow +\infty$ , we obtain

$$v^*(x) \geq \liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v]. \quad (3.3.17)$$

Let  $\delta > 0$ . By the definition of  $S_\varepsilon[x, v_*]$ , we can take  $(\tilde{p}, \tilde{\Gamma}) \in B_{\varepsilon^{-\beta}} \times B_{\varepsilon^{-\gamma}}$  which satisfies

$$\min_{\Delta x} \left[ e^{-\xi \varepsilon^2} v_*(x + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2} \langle \tilde{\Gamma} \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) \right] \geq S_\varepsilon[x, v_*] - \delta. \quad (3.3.18)$$

By considering the same choice in the definition of  $S_\varepsilon[x, v]$ , we get

$$S_\varepsilon[x_k, v] \geq \min_{\Delta x} \left[ e^{-\xi \varepsilon^2} v(x_k + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2} \langle \tilde{\Gamma} \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) \right]. \quad (3.3.19)$$

Fix  $\tilde{\delta} > 0$ . For each  $k$ , we choose  $\|\widetilde{\Delta x_k}\| \leq \varepsilon^{1-\alpha}$  such that

$$\begin{aligned} \min_{\Delta x} \left[ e^{-\xi \varepsilon^2} v(x_k + \Delta x) - \left( \tilde{p} \cdot \Delta x + \frac{1}{2} \langle \tilde{\Gamma} \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) \right] \\ \geq e^{-\xi \varepsilon^2} v(x_k + \widetilde{\Delta x_k}) - \left( \tilde{p} \cdot \widetilde{\Delta x_k} + \frac{1}{2} \langle \tilde{\Gamma} \widetilde{\Delta x_k}, \widetilde{\Delta x_k} \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) - \tilde{\delta}. \end{aligned}$$

By combining (3.3.18) and (3.3.19) and using that  $u \geq u_*$ , we deduce that

$$\liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] \geq \liminf_{k \rightarrow +\infty} e^{-\xi \varepsilon^2} v_*(x_k + \widetilde{\Delta x_k}) - \left( \tilde{p} \cdot \widetilde{\Delta x_k} + \frac{1}{2} \langle \tilde{\Gamma} \widetilde{\Delta x_k}, \widetilde{\Delta x_k} \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) - \tilde{\delta}. \quad (3.3.20)$$

After a first extraction on the sequence  $y_k = (x_k, \widetilde{\Delta x_k})$  realizing the liminf in the right-hand side of (3.3.20), we can assume that

$$\liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] \geq \lim_{k \rightarrow +\infty} e^{-\xi \varepsilon^2} v_*(x_k + \widetilde{\Delta x_k}) - \left( \tilde{p} \cdot \widetilde{\Delta x_k} + \frac{1}{2} \langle \tilde{\Gamma} \widetilde{\Delta x_k}, \widetilde{\Delta x_k} \rangle + \varepsilon^2 \mathcal{F}(\tilde{\Gamma}) \right) - \tilde{\delta}. \quad (3.3.21)$$

Then, since the sequence  $(y_k)$  lives in a compact set of  $\mathbb{R}^d \times \mathbb{R}^d$ , we can extract a subsequence converging to a point  $(\bar{x}, \widetilde{\Delta x})$ . Notice that  $\bar{x} = x$  and denote  $y = x + \widetilde{\Delta x}$ . Thus, up to this subsequence, the inequality (3.3.21) still holds.  $(\widetilde{\Delta x_k})_k$  tends to  $y - x$ . The following decomposition holds

$$\begin{aligned} e^{-\xi \varepsilon^2} v(x_k + \widetilde{\Delta x_k}) - \left( \tilde{p} \cdot \widetilde{\Delta x_k} + \frac{1}{2} \langle \tilde{\Gamma} \widetilde{\Delta x_k}, \widetilde{\Delta x_k} \rangle \right) \\ \geq e^{-\xi \varepsilon^2} v_*(y) - \left( \tilde{p} \cdot (y - x) + \frac{1}{2} \langle \tilde{\Gamma} (y - x), (y - x) \rangle \right) + e^{-\xi \varepsilon^2} (v_*(x_k + \widetilde{\Delta x_k}) - v_*(y)) \\ - \left( \tilde{p} \cdot (\widetilde{\Delta x_k} - (y - x)) + \frac{1}{2} \langle \tilde{\Gamma} (\widetilde{\Delta x_k} - (y - x)), (\widetilde{\Delta x_k} - (y - x)) \rangle \right). \end{aligned} \quad (3.3.22)$$

It is clear that

$$\begin{aligned} \left| \tilde{p} \cdot (\widetilde{\Delta x_k} - (y - x)) + \frac{1}{2} \langle \tilde{\Gamma} (\widetilde{\Delta x_k} - (y - x)), (\widetilde{\Delta x_k} - (y - x)) \rangle \right| \\ \leq \|\tilde{p}\| \|\widetilde{\Delta x_k} - (y - x)\| + \frac{1}{2} \|\tilde{\Gamma}\| \|\widetilde{\Delta x_k} - (y - x)\|^2. \end{aligned} \quad (3.3.23)$$

By combining (3.3.23) and (3.3.22) in the right-hand side of (3.3.20), we obtain

$$\begin{aligned} \liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] &\geq \liminf_{k \rightarrow +\infty} \left\{ e^{-\xi\varepsilon^2} v_\star(y) - \left( \tilde{p} \cdot (y - x) + \frac{1}{2} \langle \tilde{\Gamma}(y - x), (y - x) \rangle \right) \right. \\ &\quad \left. + e^{-\xi\varepsilon^2} (v_\star(x_k + \widetilde{\Delta x_k}) - v_\star(y)) - \|\tilde{p}\| \|\widetilde{\Delta x_k} - (y - x)\| + \frac{1}{2} \|\tilde{\Gamma}\| \|\widetilde{\Delta x_k} - (y - x)\|^2 - \tilde{\delta} \right\}. \end{aligned}$$

By using that  $x_k \rightarrow x$  and  $\widetilde{\Delta x_k} \rightarrow y - x$ , we obtain

$$\begin{aligned} \liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] &\geq e^{-\xi\varepsilon^2} v_\star(y) - \left( \tilde{p} \cdot (y - x) + \frac{1}{2} \langle \tilde{\Gamma}(y - x), (y - x) \rangle \right) \\ &\quad + \liminf_{k \rightarrow +\infty} e^{-\xi\varepsilon^2} (v_\star(x_k + \widetilde{\Delta x_k}) - v_\star(y)) - \tilde{\delta}. \end{aligned} \quad (3.3.24)$$

By using (3.3.18), we have that

$$e^{-\xi\varepsilon^2} v_\star(y) - \left( \tilde{p} \cdot (y - x) + \frac{1}{2} \langle \tilde{\Gamma}(y - x), (y - x) \rangle \right) \geq S_\varepsilon[x, v_\star] - \delta.$$

By this inequality, we deduce from (3.3.24) that

$$\liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] \geq S_\varepsilon[x, v_\star] + \liminf_{k \rightarrow +\infty} e^{-\xi\varepsilon^2} (v_\star(x_k + \widetilde{\Delta x_k}) - v_\star(y)) - (\delta + \tilde{\delta}).$$

By recalling that  $v_\star$  is lower semicontinuous, we get

$$\liminf_{k \rightarrow +\infty} S_\varepsilon[x_k, v] \geq S_\varepsilon[x, v_\star] - (\delta + \tilde{\delta}),$$

Combining this inequality with (3.3.10) and sending  $\delta, \tilde{\delta} \rightarrow 0$ , the desired estimate given by (3.3.16) holds.  $\square$

### 3.3.3.2 Consistency of the scheme with the boundary condition

Here we are going to study the behavior of the scheme approximation  $u^\varepsilon$ . It will give the rate of convergence of the scheme near the boundary. We are going to proceed by the construction of suitable barrier functions to bound  $u^\varepsilon$  and the consistency of the operator  $F_\varepsilon$  with respect to the PDE. To do this, we introduce the operator  $R_\varepsilon$  defined by

$$R_\varepsilon[x, \phi] := \sup_{p, \Gamma} \inf_{\Delta x} \left[ e^{-\xi\varepsilon^2} \phi(x + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \right], \quad (3.3.25)$$

where  $p, \Gamma$  and  $\Delta x$  are respectively constrained by (3.1.9) and (3.1.10).

**Lemma 3.3.5.** *If  $\text{dist}(x) < \varepsilon^{1-\alpha}$ , then for every  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $x \in \mathbb{R}^d$ , we have*

$$e^{-\xi\varepsilon^2} \inf_{x + \Delta x \notin U} (\hat{\phi} - \phi)(x + \Delta x) \leq S_\varepsilon[x, \phi] - R_\varepsilon[x, \phi] \leq e^{-\xi\varepsilon^2} \sup_{x + \Delta x \notin U} (\hat{\phi} - \phi)(x + \Delta x), \quad (3.3.26)$$

where  $R_\varepsilon$  is the operator defined by (3.3.25).

*Proof.* We give the proof of the first inequality, the proof of the second being strictly parallel. Observe that for all  $\|\Delta x\| \leq \varepsilon^{1-\alpha}$ , we have

$$\hat{\phi} + e^{-\xi\varepsilon^2} \inf_{\|\Delta x\| \leq \varepsilon^{1-\alpha}} (\hat{\phi} - \phi)(x + \Delta x) \leq \phi,$$

and we deduce the desired inequality by monotonicity of the operator  $S_\varepsilon$ .  $\square$

In our analysis, an important role will be played by the functions  $\psi$  which we define by

$$\psi(x) := c(e^{-b|x|^2} - e^{-b\delta^2}). \quad (3.3.27)$$

The parameters  $b, c > 0$  will be adjusted with the uniform ellipticity constants and the level set of  $\psi$  on  $\partial B_\delta$  for  $\delta > 0$ . More precisely, we will consider the choice given by

$$b := \max \left\{ 1, \frac{1 + 2d\Lambda + 2\xi}{2\delta^2\lambda} \right\}. \quad (3.3.28)$$

and

$$c := (e^{-b\delta^2} - e^{-4b\delta^2})^{-1}. \quad (3.3.29)$$

For this choice of parameters, we next show that  $\psi$  is a subsolution for the Pucci equations. Similarly, we can also construct a supersolution  $\chi$  for the same equations.

**Lemma 3.3.6.** *Let  $0 < \delta < R/2$  and  $\xi > 0$ . For a certain  $\theta > 0$ , there exists a smooth function  $\psi$  defined on  $\mathbb{R}^d$  satisfying*

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\psi) + \xi|\psi| \leq -\theta < 0, \quad \text{on } B_{2\delta} \setminus B_\delta$$

*with  $\psi > 0$  on  $B_\delta$ ,  $\psi = 0$  on  $\partial B_\delta$ ,  $\psi \leq 0$  on  $B_R \setminus B_\delta$  and  $\psi \leq -1$  on  $B_R \setminus B_{2\delta}$ . Similarly there exists a smooth function  $\chi$  defined on  $\mathbb{R}^d$  satisfying*

$$\mathcal{P}_{\lambda,\Lambda}^-(D^2\chi) - \xi|\chi| \geq \theta > 0, \quad \text{on } B_{2\delta} \setminus B_\delta$$

*with  $\chi \leq 0$  on  $B_\delta$ ,  $\chi = 0$  on  $\partial B_\delta$ ,  $\chi \geq 0$  on  $B_{2\delta} \setminus B_\delta$  and  $\chi \geq 1$  on  $B_R \setminus B_{2\delta}$ .*

*Proof.* Let us examine the first assertion, the proof for the first being strictly parallel by considering for  $\phi$  a function similar to  $\psi$ . It is immediate to check that the sign conditions are satisfied. It remains to verify the Pucci inequality. Note that for all for  $x \in B_{2\delta} \setminus B_\delta$ ,  $\psi(x) = c(e^{-b|x|^2} - e^{-b\delta^2}) \leq 0$ . Hence, we compute on  $B_{2\delta} \setminus B_\delta$  that

$$\begin{aligned} \mathcal{P}_{\lambda,\Lambda}^+(D^2\psi) + \xi|\psi| &\leq ce^{-b|x|^2} \mathcal{P}_{\lambda,\Lambda}^+(-2b(I - 2bx \otimes x)) + c\xi(e^{-b\delta^2} - e^{-b|x|^2}) \\ &\leq ce^{-b|x|^2} (2b\{(d-1)\Lambda - (2b|x|^2 - 1)\lambda\} - \xi) + c\xi e^{-b\delta^2} \\ &\leq c \left[ e^{-b|x|^2} (2bd\Lambda - 4b^2|x|^2\lambda - \xi) + \xi e^{-b\delta^2} \right] \\ &\leq c \left[ e^{-b|x|^2} (2b(d\Lambda - 2b\delta^2\lambda) - \xi) + \xi e^{-\alpha\delta^2} \right]. \end{aligned}$$

The choice (3.3.28) for  $b$  ensures us that  $2b(d\Lambda - 2b\delta^2\lambda) - \xi \leq -\xi$ . Hence

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\psi) + \xi|\psi| \leq c \left( -\xi e^{-b|x|^2} + \xi e^{-b\delta^2} \right) \leq c \left( -\xi e^{-4b\delta^2} + \xi e^{-b\delta^2} \right) \leq -\xi.$$

For the particular choice  $\theta = \xi$ , we deduce the result.  $\square$

To study  $u^\varepsilon$  near the boundary, we consider two types of barrier functions. For  $x_1 \in \mathbb{R}^d$ , define

$$\varphi_-(x) := a_- + K\psi(x - x_1) \quad \text{and} \quad \varphi_+(x) := a_+ + K\chi(x - x_1). \quad (3.3.30)$$

where the functions  $\psi, \chi$  are given by (3.3.27)–(3.3.29) and satisfy in particular the inequalities in Lemma 3.3.6 and the parameters  $a_-, a_+, K \in \mathbb{R}$  are given by

$$a_- := g(x_0) - 3\omega_g(\varepsilon^{1-\alpha}) - (2\|g\|_{L^\infty} + 1) \|D\psi\|_{L^\infty(B_\delta \setminus B_{\delta/2})} \varepsilon^{1-\alpha} \quad (3.3.31)$$

$$a_+ := g(x_0) + 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1) \|D\psi\|_{L^\infty(B_\delta \setminus B_{\delta/2})} \varepsilon^{1-\alpha} \quad (3.3.32)$$

and

$$K := 2\|g\|_{L^\infty} + 1. \quad (3.3.33)$$

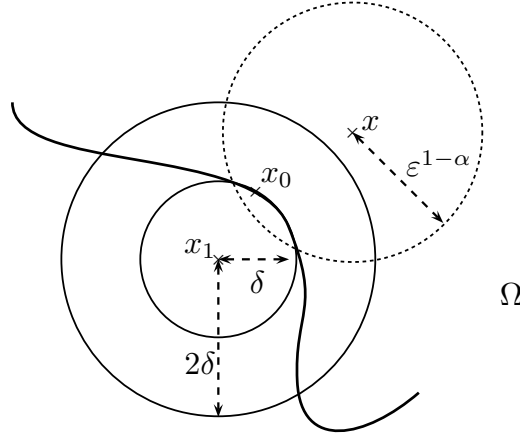


Figure 3.1: Construction of the suitable barrier functions :  $x \in U$ ,  $x_0 \in \partial U$  and  $x_1 \in \mathbb{R}^d \setminus U$ .

The regularity of the domain is here important. Since the bounded domain  $U$  has a boundary  $\partial U$  of class  $C^2$  and bounded,  $U$  satisfies the uniform exterior ball condition (see Section 1) for a certain radius ball  $\delta > 0$  sufficiently small. We impose  $\varepsilon$  to be small enough with respect to  $\delta$  by assuming that

$$\varepsilon \leq \left( \frac{\delta}{4} \right)^{\frac{1}{1-\alpha}}. \quad (3.3.34)$$

**Proposition 3.3.7.** *Let  $\varphi_-$ ,  $\varphi_+$  be defined by (3.3.30) with  $a_-$ ,  $a_+$ ,  $K \in \mathbb{R}$  respectively defined by (3.3.31), (3.3.32) and (3.3.33) and  $\psi, \chi$  given by Lemma 3.3.6. Then  $\varphi_-$  and  $\varphi_+$  satisfy respectively*

$$F_\varepsilon[\varphi_-] \leq -K \left( \frac{\theta}{2} - \|D^3\psi\|_{L^\infty} \varepsilon^{1-3\alpha} \right) \quad \text{on } U.$$

and

$$F_\varepsilon[\varphi_+] \geq K \left( \frac{\theta}{2} - \|D^3\chi\|_{L^\infty} \varepsilon^{1-3\alpha} \right) \quad \text{on } U.$$

*Proof.* The proof is divided into two steps. The first step is a preliminary to compute some inf/max for the barrier functions. Then Step 2 computes the action of  $F_\varepsilon$  on these barriers.

*Step 1.* We claim that for all  $x$  such that  $\text{dist}(x, \partial U) \leq \varepsilon^{1-\alpha}$ , we have

$$\inf_{x+\Delta x \notin U} (\hat{\varphi}_- - \varphi_-)(x + \Delta x) \geq 0, \quad (3.3.35)$$

and

$$\sup_{x+\Delta x \notin U} (\hat{\varphi}_+ - \varphi_+)(x + \Delta x) \leq 0. \quad (3.3.36)$$

We give the proof for (3.3.35), the second inequality is quite similar. Let  $x_0 \in \partial U$  such that  $\text{dist}(x_0, x) \leq \varepsilon^{1-\alpha}$ . We need to distinguish several cases. If  $x + \Delta x \in U$ ,  $(\hat{\varphi}_- - \varphi_-)(x + \Delta x) = 0$ .

If  $x + \Delta x - x_1 \in B_\delta \cap (\mathbb{R}^d \setminus U)$ , observe that by the triangle inequality

$$\|x + \Delta x - x_1\| \geq \|x_1 - x_0\| - \|x + \Delta x - x_0\| \geq \delta - \text{dist}(x, \partial U) - \|\Delta x\| \geq \delta - 2\varepsilon^{1-\alpha}.$$

Therefore since  $\psi = 0$  on  $\partial B_\delta$  by Lemma 3.3.6, and  $\varepsilon$  is chosen small enough such that (3.3.34) holds, we deduce that there exists  $C = \|D\psi\|_{L^\infty(B_\delta \setminus B_{\delta/2})} > 0$  (independent of  $\varepsilon$ ) such that

$$|\psi(x + \Delta x - x_1)| \leq C\varepsilon^{1-\alpha}.$$

Therefore we get that

$$\begin{aligned} (\hat{\varphi} - \varphi)(x + \Delta x) &= g(x + \Delta x) - a - K\psi(x + \Delta x - x_1) \\ &\geq g(x_0) - 2\omega_g(\varepsilon^{1-\alpha}) - a - KC\varepsilon^{1-\alpha}. \end{aligned}$$

If  $x + \Delta x - x_1 \in (B_{2\delta} \setminus B_\delta) \cap (\mathbb{R}^d \setminus U)$ , then  $\psi(x + \Delta x - x_1) \leq 0$  by Lemma 3.3.6. By combining this observation with the uniform continuity of  $g$  for  $\delta$  sufficiently small, we get that

$$\begin{aligned} (\hat{\varphi} - \varphi)(x + \Delta x) &= g(x + \Delta x) - a - K\psi(x + \Delta x - x_1) \\ &\geq g(x_0) - 2\omega_g(\varepsilon^{1-\alpha}) - a. \end{aligned}$$

If  $x + \Delta x - x_1 \in (B_R \setminus B_{2\delta}) \cap (\mathbb{R}^d \setminus U)$ , we get by Lemma 3.3.6 that

$$\begin{aligned} (\hat{\varphi} - \varphi)(x + \Delta x) &= g(x + \Delta x) - a - K\psi(x + \Delta x - x_1) \\ &\geq \inf_{y \in \partial U} g(y) - a + K. \end{aligned}$$

Therefore, by combining the three cases above, it suffices to choose  $a$  and  $K$  such that

$$\min \left\{ g(x_0) - 2\eta - a, g(x_0) - 2\omega_g(\varepsilon^{1-\alpha}) - a - KC\varepsilon^{1-\alpha}, \inf_{y \in \partial U} g(y) - a + K \right\} \geq 0.$$

We can choose for instance  $a = g(x_0) - 3\omega_g(\varepsilon^{1-\alpha}) - KC\varepsilon^{1-\alpha}$  and  $K = 2\|g\|_{L^\infty} + 1$ .

We now give the proof for (3.3.36). Let  $x_0 \in \partial U$  such that  $\text{dist}(x_0, x) \leq \varepsilon^{1-\alpha}$ . We need to distinguish several cases. If  $x + \Delta x \in U$ ,  $(\hat{\varphi} - \varphi)(x + \Delta x) = 0$ .

For  $x + \Delta x - x_1 \in (B_{2\delta} \setminus B_\delta) \cap (\mathbb{R}^d \setminus U)$ , notice  $\psi(x + \Delta x - x_1) \leq 0$  by Lemma 3.3.6. By combining this observation with the modulus of continuity of  $g$  for  $\delta$  sufficiently small, we get that

$$(\hat{\varphi} - \varphi)(x + \Delta x) = g(x + \Delta x) - a - K\psi(x + \Delta x - x_1) \leq g(x_0) + 2\omega_g(\varepsilon^{1-\alpha}) - a.$$

For  $x + \Delta x - x_1 \in B_R \setminus B_{2\delta}$ , we get by Lemma 3.3.6 that

$$\begin{aligned} (\hat{\varphi} - \varphi)(x + \Delta x) &= g(x + \Delta x) - a - K\psi(x + \Delta x - x_1) \\ &\leq \sup_{y \in \partial U} g(y) - a - K. \end{aligned}$$

Therefore, it suffices to choose  $a$  and  $K$  such that

$$\max \left\{ g(x_0) + 2\omega_g(\varepsilon^{1-\alpha}) - a, \sup_{y \in \partial U} g(y) - a + K \right\} \leq 0.$$

We can choose for instance  $a = g(x_0) + 3\omega_g(\varepsilon^{1-\alpha}) + C\varepsilon^{1-\alpha}$  and  $K = 2\|g\|_{L^\infty} + 1$ .

*Step 2.* We are going to prove the result for  $\varphi_-$ . First, let  $x \in U$  satisfy  $\text{dist}(x, \partial U) > \varepsilon^{1-\alpha}$ . By (S2), we have

$$F_\varepsilon[\varphi_-](x) \leq \xi a + \xi K\psi(x) + \mathcal{F}(KD^2\psi(x)) + KK_1(1 + M)\varepsilon^\nu.$$

where  $M = \|D^3\psi\|_{L^\infty}$ . By combining (F1), (F2) and the positive homogeneity of  $\mathcal{P}_{\lambda, \Lambda}^+$ , one obtains

$$\xi K\psi(x) + \mathcal{F}(KD^2\psi(x)) \leq K \left( \mathcal{P}_{\lambda, \Lambda}^+(D^2\underline{\psi}(x)) + \xi|\underline{\psi}(x)| \right),$$

By applying Lemma 3.3.6, we get that

$$\xi a + \xi K\psi(x) + \mathcal{F}(KD^2\psi(x)) \leq \xi a + K\theta, \quad (3.3.37)$$

If  $\xi a \leq 0$ , there is no condition to impose  $K$  in order to make the right-hand side of (3.3.37) be negative. Otherwise we choose  $K$  large enough  $K \geq \frac{1}{2} - \xi a/\theta$  such that the right-hand side in the inequality given by (3.3.37) is  $\leq \theta/2$ . Hence, we deduce by combining the two cases that

$$F_\varepsilon[\varphi_-](x) \leq K \left( -\frac{1}{2}\theta + K_1(1 + M)\varepsilon^\nu \right). \quad (3.3.38)$$

Assume now that  $\text{dist}(x, \partial U) \leq \varepsilon^{1-\alpha}$ . By Lemma 3.3.5, we obtain that

$$F_\varepsilon[\varphi_-](x) \leq \frac{1}{\varepsilon^2} (\varphi_-(x) - R_\varepsilon[x, \varphi_-]) - \frac{e^{-\xi\varepsilon^2}}{\varepsilon^2} \inf_{x+\Delta\hat{x} \notin U} (\hat{\varphi} - \varphi)(x + \Delta x). \quad (3.3.39)$$

Since  $\varphi_- \in C^2(B(x, 2\varepsilon^{1-\alpha}))$  for  $\varepsilon$  small enough, applying (S2) to  $\varphi$  yields

$$\frac{1}{\varepsilon^2} (\varphi_-(x) - R_\varepsilon[x, \varphi_-]) \leq K (\xi\psi(x) + \mathcal{F}(D^2\psi(x)) + K_1(1+M)\varepsilon^\nu).$$

By dividing (3.3.39) by  $\varepsilon^2$ , we deduce by using the inequality (3.3.38) that

$$F_\varepsilon[\varphi_-](x) \leq K \left( -\frac{\theta}{2} + K_1(1+M)\varepsilon^\nu \right) - \frac{e^{-\xi\varepsilon^2}}{\varepsilon^2} \inf_{x+\Delta\hat{x} \notin U} (\hat{\varphi} - \varphi_-)(x + \Delta x).$$

By applying Step 1, we deduce the desired result.

We now give the estimate for  $\varphi_+$ . If  $\text{dist}(x, \partial U) > \varepsilon^{1-\alpha}$ , the result comes directly from (S2). Assume now that  $\text{dist}(x, \partial U) \leq \varepsilon^{1-\alpha}$ . By Lemma 3.3.5, we obtain that

$$F_\varepsilon[\varphi_+](x) \geq \frac{1}{\varepsilon^2} (\varphi_+(x) - R_\varepsilon[x, \varphi_+]) + \frac{e^{-\xi\varepsilon^2}}{\varepsilon^2} \sup_{x+\Delta x \notin U} (\hat{\varphi}_+ - \varphi_+)(x + \Delta x). \quad (3.3.40)$$

Since  $\varphi_+ \in C^2(B(x, 2\varepsilon^{1-\alpha}))$  for  $\varepsilon$  small enough, applying (S2) to  $R_\varepsilon$  and  $\varphi_+$  yields

$$\frac{1}{\varepsilon^2} (\varphi_+(x) - R_\varepsilon[x, \varphi_+]) \geq K (\xi\psi_+(x) + \mathcal{F}(D^2\psi_+(x)) - K_1(1+M)\varepsilon^\nu).$$

By dividing (3.3.40) by  $\varepsilon^2$ , we deduce by Step 1 that

$$F_\varepsilon[\varphi_+](x) \geq K \left( \frac{\theta}{2} - K_1(1+M)\varepsilon^\nu \right) + \frac{e^{-\xi\varepsilon^2}}{\varepsilon^2} \sup_{x+\Delta x \notin U} (\hat{\varphi}_+ - \varphi_+)(x + \Delta x).$$

By applying Step 1, we deduce the desired result.  $\square$

**Proposition 3.3.8** (Consistency of the game with the boundary data). *Assume that  $\mathcal{F}$  satisfies (F1) and  $u_\varepsilon$  is the solution of (3.1.11). Then, for  $\varepsilon$  small enough, for every  $x \in U$  and  $x_0 \in \partial U$ , we have*

$$\begin{aligned} -3\omega_g(\varepsilon^{1-\alpha}) - (2\|g\|_{L^\infty} + 1) [ \|D\psi\|_{L^\infty(B_{2\delta} \setminus B_\delta)} \varepsilon^{1-\alpha} + \psi(x - x_1) ] &\leq u_\varepsilon(x) - g(x_0) \\ &\leq 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1) [ \|D\chi\|_{L^\infty(B_{2\delta} \setminus B_\delta)} \varepsilon^{1-\alpha} + \chi(x - x_1) ]. \end{aligned}$$

*Proof.* We are going to prove the first inequality, the arguments for the second inequality being the same by considering  $\varphi_+$ . Fix  $x_0 \in \partial U$ . We consider the function  $\varphi_-$  defined by (3.3.30) for a certain  $x_1 \in \mathbb{R}^d \setminus U$  such that  $x_0 \in \partial B_\delta(x_1)$ . According to Proposition 3.3.7,  $\varphi_-$  is a  $\varepsilon$ -discrete subsolution since

$$F_\varepsilon[\varphi_-] \leq 0 \quad \text{on } U.$$

Then by using (S1G), we get

$$\varphi_- \leq u_\varepsilon \quad \text{on } U.$$

which precisely gives the first inequality.  $\square$

To prove the Lipschitz estimate on the scheme given by Proposition 3.1.2 item (P3), it is sufficient to prove the following proposition.

**Proposition 3.3.9.** *Assume that  $\text{diam } U < +\infty$ . For all  $x, y \in U$ , we have*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq e^{-\xi\varepsilon^2} \{ 3K\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1)C(K\varepsilon^{1-\alpha} + \text{dist}(x, y)) \},$$

where  $K = \lfloor \frac{\text{dist}(x, y)}{\varepsilon^{1-\alpha}} \rfloor + 1$ .



*Proof. Step 1.* For all  $x, y \in U$ , we have

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq e^{-\xi\varepsilon^2} \|\hat{u}_\varepsilon(\cdot) - \hat{u}_\varepsilon(\cdot + y - x)\|_{L^\infty(\overline{U})}.$$

To prove it, the dynamic programming principle gives

$$u_\varepsilon(x) = \max_{p, \Gamma} \min_{\Delta x} \left[ e^{-\xi\varepsilon^2} \hat{u}_\varepsilon(x + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \right]$$

If we hold  $p, \Gamma$  and  $\Delta x$  fixed, it is clear that

$$\begin{aligned} e^{-\xi\varepsilon^2} \hat{u}_\varepsilon(x + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \\ \leq e^{-\xi\varepsilon^2} \hat{u}_\varepsilon(y + \Delta x) - \left( p \cdot \Delta x + \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle + \varepsilon^2 \mathcal{F}(\Gamma) \right) \\ + \varepsilon^{-\xi\varepsilon^2} \sup_{\|\Delta x\| \leq \varepsilon^{1-\alpha}} |\hat{u}_\varepsilon(x + \Delta x) - \hat{u}_\varepsilon(y + \Delta x)|. \end{aligned}$$

By minimizing over  $\Delta x$  and then by maximizing over  $p$  and  $\Gamma$  we find that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \varepsilon^{-\xi\varepsilon^2} \sup_{\|\Delta x\| \leq \varepsilon^{1-\alpha}} |\hat{u}_\varepsilon(x + \Delta x) - \hat{u}_\varepsilon(y + \Delta x)|,$$

and we get the claim by noticing that

$$\sup_{\|\Delta x\| \leq \varepsilon^{1-\alpha}} |\hat{u}_\varepsilon(x + \Delta x) - \hat{u}_\varepsilon(y + \Delta x)| \leq \|\hat{u}_\varepsilon(\cdot) - \hat{u}_\varepsilon(\cdot + y - x)\|_{L^\infty(\overline{U})}.$$

*Step 2.* We claim that there exists a constant  $C > 0$  such that for all  $a \in \mathbb{R}^d$ , we have

$$\|\hat{u}_\varepsilon(\cdot) - \hat{u}_\varepsilon(\cdot + a)\|_{L^\infty(\overline{U})} \leq \max \{ \omega_g(|a|), 3\omega_g(\varepsilon^{1-\alpha}) + C(2\|g\|_{L^\infty} + 1)(\varepsilon^{1-\alpha} + |a|) \}.$$

We may assume that the sup in the l.h.s. is positive, otherwise the result is clear. By considering the different cases whether  $x$  and  $x + a$  are or not in  $U$ , the following decomposition holds

$$\begin{aligned} \|\hat{u}_\varepsilon(\cdot) - \hat{u}_\varepsilon(\cdot + a)\|_{L^\infty(\overline{U})} = \max \left\{ \sup_{y \in U} |u_\varepsilon(y) - \hat{u}_\varepsilon(y + a)|, \right. \\ \left. \sup_{y \notin U, y+a \in U} |g(y) - u_\varepsilon(y + a)|, \sup_{y \notin U, y+a \notin U} |g(y) - g(y + a)| \right\}. \quad (3.3.41) \end{aligned}$$

First, we show that the first alternative cannot happen. Arguing by contradiction, assume that the max corresponds to the first alternative. Now pick a sequence  $x_k \in U$  such that

$$|u_\varepsilon(x_k) - \hat{u}_\varepsilon(x_k + a)| \xrightarrow{k \rightarrow +\infty} \sup_{y \in U} |u_\varepsilon(y) - \hat{u}_\varepsilon(y + a)|.$$

Observe that

$$|u_\varepsilon(x_k) - \hat{u}_\varepsilon(x_k + a)| = \begin{cases} |u_\varepsilon(x_k) - u_\varepsilon(x_k + a)|, & \text{if } x_k + a \in U, \\ |u_\varepsilon(x_k) - g(x_k + a)|, & \text{if } x_k + a \notin U. \end{cases} \quad (3.3.42)$$

If the sup corresponds to the second alternative in (3.3.42), then we can conclude by Proposition 3.3.8. Actually there exists a constant  $C$  independent of  $\varepsilon$  such that, for all  $k$ ,

$$|u_\varepsilon(x_k) - g(x_k + a)| \leq 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1)C(\varepsilon^{1-\alpha} + |a|).$$

By taking the limit  $k \rightarrow +\infty$ , we get the desired result. Assume now that the sup corresponds to the first alternative. Up to extracting a subsequence, we may consider that  $x_k + a \in U$ . By Step 1, we get that for all  $k$  large enough

$$|u_\varepsilon(x_k) - u_\varepsilon(x_k + a)| \leq e^{-\xi\varepsilon^2} \|\hat{u}_\varepsilon(\cdot) - \hat{u}_\varepsilon(\cdot + a)\|_{L^\infty(\overline{U})} = e^{-\xi\varepsilon^2} \sup_{y \in U} |u_\varepsilon(y) - \hat{u}_\varepsilon(y + a)|.$$

By taking the limit  $k \rightarrow +\infty$ , we get a contradiction since  $0 < e^{-\xi\varepsilon^2} < 1$  and the first alternative can never happen. Then by definition of the modulus of continuity of  $g$ , we know that

$$\sup_{y \notin U, y+a \notin U} |g(y) - g(y+a)| \leq \omega_g(|a|).$$

By Proposition 3.3.8, there exists a constant  $C$  independent of  $\varepsilon$  such that,

$$\sup_{y \notin U, y+a \in U} |g(y) - u_\varepsilon(y+a)| \leq 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1)C(\varepsilon^{1-\alpha} + |a|).$$

By combining the two last inequalities in (3.3.41), we deduce the claim.

*Step 3.* Let  $x, y \in U$ . We pick a path  $\{x_0 = x, x_1, \dots, x_K = y\}$  linking  $x$  and  $y$  satisfying

$$\begin{cases} \frac{1}{2}\varepsilon^{1-\alpha} \leq \text{dist}(x_i, x_{i+1}) \leq \varepsilon^{1-\alpha}, & \text{for all } 0 \leq i \leq K-2, \\ \text{dist}(x_{K-1}, x_K) \leq \varepsilon^{1-\alpha}, \end{cases}$$

and

$$\text{dist}(x, y) \leq \sum_{i=0}^{K-1} \text{dist}(x_i, x_{i+1}) = \sum_{i=0}^{K-1} |x_i - x_{i+1}| \leq \text{dist}(x, y) + \frac{1}{2}\varepsilon^{1-\alpha},$$

where  $K = \lfloor \frac{\text{dist}(x, y)}{\varepsilon^{1-\alpha}} \rfloor + 1$ . We can make this decomposition since  $U$  is smooth and  $\text{diam } U < +\infty$ . By using Step 2, we get that

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(y)| &\leq \sum_{i=0}^{K-1} |u_\varepsilon(x_i) - u_\varepsilon(x_{i+1})| \\ &\leq e^{-\xi\varepsilon^2} \sum_{i=0}^{K-1} \max(\omega_g(d(x_{i+1}, x_i)), 3\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1)C(\varepsilon^{1-\alpha} + |x_{i+1} - x_i|)). \end{aligned}$$

By monotonicity of  $\omega_g$  and construction of the path, we obtain that

$$e^{\xi\varepsilon^2} |u_\varepsilon(x) - u_\varepsilon(y)| \leq 3K\omega_g(\varepsilon^{1-\alpha}) + (2\|g\|_{L^\infty} + 1) \left( KC\varepsilon^{1-\alpha} + \sum_{i=0}^{K-1} |x_{i+1} - x_i| \right)$$

Since  $(x_i)_i$  is a geodesic path between  $x$  and  $y$ , we obtain the desired inequality.  $\square$

## 3.4 Proof of Proposition 3.2.2

In this section we give the argument for the double-variable maximum principle, Proposition 3.2.2. The proof is based on ABP-type estimates in both the continuous and scheme settings. We state the latter in the following two lemmas. In addition, we need to use a measure argument using the space time step  $\varepsilon$  of the scheme.

**Lemma 3.4.1.** *Suppose that  $U \subseteq \mathbb{R}^d$ ,  $R > 0$ ,  $B_{4R} \subseteq U$  and  $u \in C(U)$  satisfies*

$$\mathcal{P}_{1,\Lambda}^+(D^2u) \geq -1 \quad \text{in } U.$$

*Assume that  $x_0, x_1 \in B_R \cap \{u = \Gamma_u\}$  and  $p_i \in \partial\Gamma_u(x_i)$  for  $i = 0, 1$ . Then there exists  $C(d, \Lambda) > 1$  such that*

$$|p_1 - p_0|^2 \leq C|x_1 - x_0|^2.$$

**Lemma 3.4.2.** Suppose that  $\mathcal{F}$  satisfies (F1)–(F2),  $U \subseteq \mathbb{R}^d$ ,  $R > 1$ ,  $B_{4R} \subset U$  and  $u : U \rightarrow \mathbb{R}$  satisfies

$$F_\varepsilon[u] \geq -1 \quad \text{in } \Lambda.$$

Assume, for each  $i \in \{0, 1\}$ , that  $z_i \in B_R \cap \{u = \Gamma_u\}$  and  $p_i \in \partial\Gamma_u(z_i)$ . Then there exists  $C(d, \Lambda, K_0, K_1) > 1$  such that  $R \geq C\varepsilon$ , implies that

$$|p_1 - p_0| \leq C(\varepsilon^{1-\alpha} + |z_1 - z_0|).$$

**Lemma 3.4.3.** Let  $H > 0$ ,  $m \geq 1$ ,  $E \subseteq \mathbb{R}^m$  be bounded and  $\phi : \mathbb{R}^m \mapsto \mathbb{R}^m$  such that for all  $x, y \in E$ ,

$$|\phi(x) - \phi(y)| \leq L|x - y| + H.$$

There exists a constant  $C > 1$  such that, if we denote  $\tilde{E}_\rho := B_\rho + E$  for  $\rho > 0$ , then

$$|\phi(E)| \leq C \left( L + \frac{H}{\rho} \right)^m |\tilde{E}_\rho|.$$

**Proof of Proposition 3.2.2.** Suppose that, for each  $i \in \{1, 2\}$ ,

$$(x_i, y_i, p_i, q_i) \in U \times U \times \mathbb{R}^d \times \mathbb{R}^d$$

satisfies

$$\Phi_{p_i, q_i}(x_i, y_i) = \sup \{ \Phi_{p_i, q_i}(\xi, \eta) : (\xi, \eta) \in U \times U \}. \quad (3.4.1)$$

Denote

$$r := (|x_1 - x_2|^2 + |y_1 - y_2|^2)^{1/2}.$$

The main step in the argument is to show that, for some  $C(d, \Lambda, K_0) > 1$ ,

$$|p_1 - p_2|^2 + |q_1 - q_2|^2 \leq C\delta^{-2}(\varepsilon^{2-2\alpha} + r^2). \quad (3.4.2)$$

The first two steps contain the proof of (3.4.2) and in the final step we deduce the proposition from (3.4.2). Throughout we let  $C$  and  $c$  denote positive constants which may vary from line to line and depend only on  $(d, \Lambda, K_0)$ .

*Step 1.* We claim that

$$|p_1 - p_2|^2 \leq C\delta^{-2}r^2. \quad (3.4.3)$$

Define

$$\tilde{u}(x) := -u(x) + \frac{1}{2\delta}|x - y_1|^2.$$

By (4.2.2) and Step 1, we find that

$$\begin{aligned} \mathcal{P}_{1, \Lambda}^+(D^2\tilde{u}) &= \mathcal{P}_{\lambda, \Lambda}^+ \left( -D^2u + \frac{1}{\delta}I_d \right) \\ &\geq -C\delta^{-1}. \end{aligned} \quad (3.4.4)$$

According to (4.3.1) with  $i = 1$ , we have

$$\tilde{u}(x_1) = \Gamma_{\tilde{u}}(x_1) \quad \text{and} \quad (-p_1, \tilde{u}(x_1) + p_1 \cdot x_1) \in \partial\Gamma_{\tilde{u}}(x_1). \quad (3.4.5)$$

We next check that (4.3.1) with  $i = 2$  implies

$$\left( -p_2 + \frac{y_2 - y_1}{\delta}, \tilde{u}(x_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \in \partial\Gamma_{\tilde{u}}(x_2). \quad (3.4.6)$$

To see this, note that the former implies that the map

$$x \mapsto u(x) - \frac{1}{2\delta}|x - y_2|^2 - p_2 \cdot x$$

achieves its supremum over  $U$  at  $x_2$ , and we may write this as

$$u(x) - \frac{1}{2\delta}|x - y_2|^2 - p_2 \cdot x = -\tilde{u}(x) - \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x + \frac{1}{2\delta}(|y_1|^2 - |y_2|^2).$$

that is,

$$x \mapsto \tilde{u}(x) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x \text{ attains its infimum on } U \text{ at } x_2.$$

This yields (4.3.11). Applying Lemma 3.4.1 in view of (4.3.15), (4.3.10) and (4.3.11) yields

$$\left|p_1 - p_2 + \frac{y_2 - y_1}{\delta}\right| \leq C\delta^{-1}|x_1 - x_2|, \quad (3.4.7)$$

*Step 2.* We show that

$$|q_1 - q_2|^2 \leq C\delta^{-2}(\varepsilon^2 + \delta + r^2). \quad (3.4.8)$$

Define

$$\tilde{v}(y) := v(y) + \frac{1}{2\delta}|y - x_1|^2.$$

According to (S3), we have

$$F_\varepsilon[\tilde{v}] \geq F_\varepsilon[v] - C\delta^{-1}.$$

According to (4.3.1) with  $i = 1$ , we have

$$\tilde{v}(y_1) = \Gamma_{\tilde{v}}(y_1) \quad \text{and} \quad (-q_1, \tilde{v}(y_1) + q_1 \cdot y_1) \in \partial\Gamma_{\tilde{v}}(y_1). \quad (3.4.9)$$

We next check that (4.3.1) for  $i = 2$  implies that

$$\left(-q_2 - \frac{x_1 - x_2}{\delta}, \tilde{v}(y_2) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y_2\right) \in \partial\Gamma_{\tilde{v}}(y_2). \quad (3.4.10)$$

We observe that

$$y \mapsto v(y) + \frac{1}{2\delta}|y - x_2|^2 + q_2 \cdot y$$

achieves its infimum over  $U$  at  $y_2$  and thus, using the identity

$$v(y) + \frac{1}{2\delta}|y - x_2|^2 + q_2 \cdot y = \tilde{v}(y) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y + \frac{1}{2\delta}(|x_2|^2 - |x_1|^2)$$

that is,

$$y \mapsto \tilde{v}(y) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y \text{ attains its infimum over } U \text{ at } y_2.$$

Applying Lemma 3.4.2 in view of (4.3.19) and (4.3.20) yields

$$\left|q_1 - q_2 + \frac{x_2 - x_1}{\delta}\right|^2 \leq C\delta^{-1}(\varepsilon^{1-\alpha} + |y_1 - y_2|)$$

By doing similar computations than in Step 2, we get

$$|q_1 - q_2| \leq C\delta^{-1}(\varepsilon^{1-\alpha} + r).$$

*Step 3.* By combining Steps 1-2, we obtain (3.4.2). We get that

$$\begin{aligned} (|p_1 - p_2|^2 + |q_1 - q_2|^2)^{1/2} &\leq C\delta^{-1}(\varepsilon^{2-2\alpha} + r^2)^{1/2} \\ &\leq C\delta^{-1}(\varepsilon^{1-\alpha} + r). \end{aligned}$$

Applying Lemma 4.3.3 yields the desired measure inequality. This completes the proof.  $\square$

**Proof of Lemma 3.4.1.** Throughout,  $C$  and  $c$  denote positive constants which may change in each occurrence and depend only on  $d$  and  $\Lambda$ . By a covering argument and rescaling, it suffices to consider the case  $R = 2$  and  $0 < r \leq c$ . Denote  $r = |x_1 - x_0|$ , we want to prove

$$|p_0 - p_1| \leq Cr. \quad (3.4.11)$$

We prove (3.4.11). We may assume by translating that  $x_1 = 0$  and, by subtracting a plane from  $u$ , that  $p_1 = 0$  and  $u(x_1) = 0$ . In that case,  $u(x_0) \geq 0$  and we have

$$u(x) \geq \Gamma_u(x) \geq \max \{0, u(x_0) - p_0 \cdot x_0 + p_0 \cdot x\} \quad \text{for } x \in B_4.$$

It is convenient to take  $e := p_0/|p_0|$ . With  $A, \sigma > 1$  to be selected below, we consider the test function

$$\varphi(x) := \frac{1}{2}Ar(x \cdot e) + \frac{1}{2}\sigma(x \cdot e)^2 - \frac{3}{2}|x - (x \cdot e)e|^2.$$

After a computation, we find that the choice  $\sigma := 1 + 3\Lambda(d-1) \leq C$  leads to

$$\mathcal{P}_{1,\Lambda}^+(D^2\varphi) = 3\Lambda(d-1) - \sigma = -1 \quad \text{in } \mathbb{R}^d.$$

The comparison principle applied to  $u$  and  $\varphi$  with respect to the domain

$$\tilde{Q} := \{x \in \mathbb{R}^d : |x - (x \cdot e)e| \leq 1, |x \cdot e| \leq \beta r\} \subseteq B_R$$

with  $\beta > 1$  to be selected below, yields

$$\inf_{\partial\tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial\tilde{Q}} (u - \varphi) = \inf_{\tilde{Q}} (u - \varphi) \leq (u - \varphi)(0) = 0.$$

Select  $y \in \partial\tilde{Q}$  such that  $(\Gamma_u - \varphi)(y) \leq 0$ . There are three possibilities to consider: (i)  $y \cdot e = \beta r$ , (ii)  $y \cdot e = -\beta r$ , (iii)  $|y - (y \cdot e)e| = 1$ . We argue that, if  $\beta$  and  $A$  are chosen appropriately, then cases (ii) – (iii) are impossible and case (i) leads to the bound (3.4.11).

In case (i), we find

$$(\beta - 1)|p_0|r \leq u(x_0) - p_0 \cdot x_0 + |p_0|\beta r \leq \Gamma_u(y) \leq \varphi(y) = \varphi(y) \leq \frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2.$$

Dividing by  $(\beta - 1)r$  and rearranging, we get

$$|p_0| \leq \frac{1}{2} \frac{A}{\beta - 1} r + \frac{1}{2} \frac{\sigma\beta}{\beta - 1} r \leq C(1 + A + \beta)r.$$

Choosing  $\beta := C$  yields  $|p_0| \leq C(1 + A)r \leq Cr$ , as desired, provided that  $A \leq C$ . Case (ii) is impossible if we select  $A := 2\sigma\beta \leq C$ , since it implies

$$0 \leq \Gamma_u(y) \leq \varphi(y) = \varphi(y) \leq -\frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2 < 0.$$

Case (iii) is also impossible if  $0 < r \leq c$ , since it implies

$$0 \leq \Gamma_u(y) \leq \varphi(y) \leq \frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2 - \frac{3}{2} \leq Cr^2 - \frac{3}{2} < 0.$$

This completes the proof of (3.4.11) and the lemma.  $\square$

We next give the proof of Lemma 3.4.2 which is an adaptation of the proof of the previous lemma to the scheme case, using an approximation comparison principle and (S2) to justify the test function computations. At the risk of repetition, we give the complete details of the argument for reader's convenience.

*Proof of Proposition 3.4.2.* Throughout,  $C$  and  $c$  denote positive constants which may change in each occurrence and depend only on  $d$  and  $\Lambda$ . By a covering argument and rescaling, it suffices to consider the case  $R = 2$  and  $0 < r \leq c$ . Denote  $r = |x_1 - x_0|$ , we want to prove

$$|p_0 - p_1| \leq C(\varepsilon^{1-\alpha} + r). \quad (3.4.12)$$

We prove (3.4.12). We may assume by translating that  $x_1 = 0$  and, by subtracting a plane from  $u$ , that  $p_1 = 0$  and  $u(x_1) = 0$ . In that case,  $u(x_0) \geq 0$  and we have

$$u(x) \geq \Gamma_u(x) \geq \max\{0, u(x_0) - p_0 \cdot x_0 + p_0 \cdot x\} \quad \text{for } x \in B_4.$$

It is convenient to take  $e := p_0/|p_0|$ . With  $A, \sigma > 1$  to be selected below, we consider the test function

$$\varphi(x) := \frac{1}{2}A(r + \varepsilon^{1-\alpha})(x \cdot e) + \frac{1}{2}\sigma(x \cdot e)^2 - \frac{3}{2}|x - (x \cdot e)e|^2.$$

We take  $\sigma := 1 + 3\Lambda(d-1) \leq C$  and apply (S2) and  $\mathcal{F}(0) = 0$  to get

$$F_\varepsilon[\varphi] = \mathcal{F}(D^2\varphi) \leq \mathcal{P}_{1,\Lambda}^+(D^2\varphi) = 3\Lambda(d-1) - \sigma = -1 \quad \text{in } \mathbb{R}^d.$$

The comparison principle applied to  $u$  and  $\varphi$  with respect to the domain

$$\tilde{Q} := \{x \in \mathbb{R}^d : |x - (x \cdot e)e| \leq R, |x \cdot e| \leq \beta r\} \subseteq B_{2R}$$

with  $\beta > 1$  to be selected below, yields

$$\inf_{\partial\tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial\tilde{Q}} (u - \varphi) = \inf_{\tilde{Q}} (u - \varphi) \leq (u - \varphi)(0) = 0.$$

Select  $y \in \partial\tilde{Q}$  such that  $(\Gamma_u - \varphi)(y) \leq 0$ . There are three possibilities to consider: (i)  $y \cdot e = \beta r$ , (ii)  $y \cdot e = -\beta r$ , (iii)  $|y - (y \cdot e)e| = 1$ . We argue that, if  $\beta$  and  $A$  are chosen appropriately, then cases (ii) – (iii) are impossible and case (i) leads to the bound (3.4.12).

In case (i), we find

$$(\beta - 1)|p_0|r \leq u(x_0) - p_0 \cdot x_0 + |p_0|\beta r \leq \Gamma_u(y) \leq \varphi(y) = \varphi(y) \leq \frac{1}{2}A\beta(r + \varepsilon^{1-\alpha})^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon^{1-\alpha})^2.$$

Dividing by  $(\beta - 1)r$ , using that  $\varepsilon^{1-\alpha} \leq r$  and rearranging, we get

$$|p_0| \leq \frac{1}{2} \frac{A}{\beta - 1} r + \frac{1}{2} \frac{\sigma\beta}{\beta - 1} r \leq C(1 + A + \beta)r.$$

Choosing  $\beta := C$  yields  $|p_0| \leq C(1 + A)r \leq Cr$ , as desired, provided that  $A \leq C$ . Case (ii) is impossible if we select  $A := 2\sigma\beta \leq C$ , since it implies

$$0 \leq \Gamma_u(y) \leq \varphi(y) = \varphi(y) \leq -\frac{1}{2}A\beta(r + \varepsilon^{1-\alpha})^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon^{1-\alpha})^2 < 0.$$

Case (iii) is also impossible if  $0 < r \leq c$ , since it implies

$$0 \leq \Gamma_u(y) \leq \varphi(y) \leq \frac{1}{2}A\beta(r + \varepsilon^{1-\alpha})^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon^{1-\alpha})^2 - \frac{3}{2}R^2 \leq Cr^2 - \frac{3}{2} < 0.$$

This completes the proof of (3.4.12) and the lemma.  $\square$

### 3.5 Appendix

In this appendix, we gather the proof of the consistency estimates and the consistency of the viscosity solution of the PDE with the boundary condition under the assumption of uniform ellipticity of the operator.

In this appendix, we gather the proof of the consistency of the viscosity solution of the PDE with the boundary condition under the assumption of uniform ellipticity of the operator.

*Proof of (3.2.4).* The proof is classical but we recall it for sake of completeness. Let  $x_0 \in \partial U$ , it suffices to show that  $u(x_0) \geq g(x_0)$ , the proof for  $u(x_0) \leq g(x_0)$  being strictly parallel. By Perron's method, the viscosity solution can be characterized by the representation formula

$$u := \sup\{v \in \text{USC}(\overline{U}) : \mathcal{F}(D^2v) \leq 0 \text{ in } U, v \leq g \text{ on } \partial U\}. \quad (\text{B.1})$$

We are going to construct a suitable subsolution of (3.1.1). Denote  $\psi$  the barrier function given by Lemma 3.3.6. Let  $\varepsilon > 0$ , we introduce the test function (with  $\delta_0 = 0$  in Lemma 3.3.6)

$$\phi(x) = g(x_0) - \varepsilon + K\psi(x - x_1) \quad \text{in } U,$$

where  $x_1 \notin U$ ,  $B_\delta(x_1) \cap \overline{U} = \{x_0\}$  such that  $R > 0$  is large enough such that  $U \subset B_R(x_1)$ . It is immediate that  $\phi(x_0) = g(x_0) - \varepsilon \leq g(x_0)$ . By using (F1) and (F2), we have

$$\mathcal{F}(D^2\phi(x)) \leq K(\mathcal{P}_{\lambda,\Lambda}^+(D^2\psi(x)) + \mu|D\psi(x)|).$$

By Lemma 3.3.6, we deduce that for any  $K$ , we have

$$\mathcal{F}(D^2\phi(x)) < 0 \quad \text{in } U.$$

Let us examine the boundary condition. On one hand, if  $K$  is chosen sufficiently large enough such that  $K \geq 2\|g\|_{L^\infty} + 1$ , we have by using Lemma 3.3.6 that

$$\phi \leq g \quad \text{on } \partial U \setminus B_{2\delta}(x_1).$$

On the other hand, we get by using the uniform continuity of  $g$  for  $\delta$  sufficiently small and using the condition in Lemma 3.3.6 that

$$\phi \leq g \quad \text{in } \partial U \cap B_{2\delta}.$$

By putting together the two cases and recalling that  $u$  can be represented by (B.1), we obtain that  $u(x_0) \geq g(x_0) - \varepsilon$ . By taking the limit  $\varepsilon \rightarrow 0$ , we get that  $u(x_0) \geq g(x_0)$ .  $\square$

# Rates of convergence for solutions of solutions of finite-difference schemes to fully nonlinear, uniformly elliptic, parabolic equations

Nous étudions le taux de convergence pour les solutions d'une classe de schémas par différences finies associée à une équation parabolique pour un opérateur non-linéaire supposé uniformément elliptique. Nous avons considéré des schémas munis d'une discrétisation implicite en temps. La méthode proposée pour obtenir un taux algébrique de convergence pour la solution du schéma vers la solution de viscosité de l'équation combine l'utilisation de l'estimation  $W^{3,\sigma}$  parabolique avec des estimations d'Aleksandrov-Bakelman-Pucci portant sur l'enveloppe monotone. Dans un travail actuellement en préparation, nous traiterons le cas d'une discrétisation explicite.

*Le travail présenté dans ce chapitre est le fruit d'une collaboration avec Scott Armstrong.*

Preprint
----------





## Chapter 4

# Rates of convergence for solutions of finite-difference schemes to fully nonlinear, uniformly elliptic, parabolic equations

### Abstract

We study an algebraic rate of convergence for solution of finite-difference schemes associated to a parabolic equation for a nonlinear uniformly elliptic operator. In this work, we focus on implicit in time discretizations. The proof of the rate of convergence mainly relies on a parabolic  $W^{3,\sigma}$  estimate and suitable Aleksandrov-Bakelman-Pucci inequalities.

**Keywords:** rates of convergence, fully nonlinear parabolic equations, viscosity solutions.

**Mathematics Subject Classification** 35B65, 35K55, 35D40, 65M06, 65M15.

### Contents

---

<b>4.1</b>	<b>Introduction</b>	<b>168</b>
4.1.1	Motivation and statement of the results	168
4.1.2	Hypotheses and notation	169
<b>4.2</b>	<b>Proof of the implicit rate of convergence</b>	<b>171</b>
4.2.1	The $W^{3,\sigma}$ estimate	171
4.2.2	A comparison principle for implicit schemes	172
4.2.3	A maximum principle in the doubled variables	173
4.2.4	Proof of the main theorem	173
<b>4.3</b>	<b>Proof of the maximum principle in the doubled variables</b>	<b>182</b>
<b>4.4</b>	<b>Appendix A: continuous version of the doubling lemma</b>	<b>191</b>
<b>4.5</b>	<b>Appendix B: Proof of Proposition 4.2.5</b>	<b>196</b>

---

## 4.1 Introduction

### 4.1.1 Motivation and statement of the results

We consider finite-difference schemes associated to fully nonlinear parabolic equations of the form

$$\partial_t u + \mathcal{F}(D^2 u) = 0 \quad \text{in } \Omega = \mathbb{R}^d \times (-1, +\infty). \quad (4.1.1)$$

associated to the initial condition

$$u(\cdot, 0) = g \quad \text{on } \mathbb{R}^d, \quad (4.1.2)$$

where the operator  $\mathcal{F}$  is assumed to be uniformly elliptic and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  a smooth function with compact support. We write  $u$  as a function of  $(x, t) \in \mathbb{R}^d \times [-1, \infty)$  and  $D^2 u$  denotes the Hessian of  $u$  with respect to the  $x$  variable. The schemes are built on a space-time cartesian grid having the form  $\Gamma_\varepsilon := \varepsilon \mathbb{Z}^d \times \varepsilon^2 \mathbb{Z}$ . The implicit scheme is given by

$$\begin{cases} \hat{\partial}_t^\varepsilon v(x, t) + F_\varepsilon[v(\cdot, t)](x) = 0, & \text{on } \Gamma_\varepsilon \cap \{(x, t) : t \geq \varepsilon^2\}, \\ v(\cdot) = g, & \text{on } \partial_p \Gamma_\varepsilon, \end{cases} \quad (4.1.3)$$

where the discrete time derivative operator  $\hat{\partial}_t^\varepsilon$  denotes

$$\hat{\partial}_t^\varepsilon v(x, t) := \frac{1}{\varepsilon^2} (v(x, t) - v(x, t - \varepsilon^2)).$$

The approximation theory developed by Barles and Souganidis [20] provides the following criteria for the convergence of approximation schemes: monotone, consistent, and stable schemes converge to the unique viscosity solution of a degenerate elliptic or parabolic equation. But this work does not indicate a quantization of the error.

The study of the convergence and error estimates for monotone and consistent approximations to fully nonlinear, first-order PDEs were established by Crandall and Lions [43] and Souganidis [133]. The convergence, without error, of monotone and consistent approximations for fully nonlinear, possibly degenerate second-order equations PDEs was first obtained by Barles and Souganidis [20]. For finite difference schemes schemes Kuo and Trudinger [98, 99] also investigated in great detail at the issues of regularity and existence of such approximations for uniformly elliptic equations. Finding a rate of convergence has been the lack of appropriate regularizations of viscosity solutions yielding control on derivatives higher than 2 except for either convex or concave  $\mathcal{F}$ .

The first result in the convex/concave case, the exponent of  $1/27$  for  $\varepsilon$ , was obtained by Krylov [93, 94] using the stochastic control interpretation of the equation that is available in the convex/concave setting. Later Barles and Jakobsen [15, 16] improved the error exponent to  $1/5$  by purely PDEs techniques using switching-control-type approximations – once again the convexity/concavity of the operator  $\mathcal{F}$  plays a crucial role. More recently, Krylov [95], always in the convex/concave but degenerate case, improved the exponent rate to  $1/2$  again using stochastic control considerations.

The main difficulty to get an error estimate between viscosity solution and approximate solutions, in both the elliptic and parabolic settings, is overcoming the lack of regularity of the viscosity solution  $u$ . The proof of the error estimate in [35] is based on the  $W^{3,\varepsilon}$  estimate, which says that outside a set of small measure, the solution of a uniformly elliptic equation has second-order expansions with controlled cubic term. In their article, Caffarelli and Souganidis strongly use the notion of inf-, sup- convolution introduced first by Lasry and Lions [100] and then by Jensen [86]. This error estimate is also a key step in getting a rate for homogenization in random media [36].

In this paper, we aim to give an alternative strategy for studying the rate of convergence of fully nonlinear parabolic equations that we believe to be of interest beyond the proof of the rate of convergence associated to (4.1.3). Here we propose a new proof based on ABP inequalities and a measure argument.

We study the error by examining the maximum points of a family functions on doubled variables which are indexed by some small slopes. Rather than introducing some inf-/sup- regularizations, a key point of our argument relies on the analysis of monotone envelopes and some Aleksandrov-Bakelman-Pucci (ABP) inequalities for both the viscosity solution and the approximate solution. The other tool is the regularity theorem provided by the parabolic  $W^{3,\sigma}$  estimate [51].

We expect our strategy to be applied on another parabolic schemes, as for instance explicit in times, under a suitable assumption of monotonicity to ensure us a comparison principle. This work is currently in preparation. We assume here that the nonlinearity depends only on the hessian matrix but other dependencies can be considered. On one hand, the uniform ellipticity assumption can be relaxed to a gradient term  $Du$ . On the other hand, it could be interesting to solve a parabolic problem with a nonlinearity depending on the  $x, t$ -variables.

We require the initial data  $g$  to be continuous, not identically zero and to have compact support. We now state our main result about implicit schemes.

**Theorem 4.1.1.** *Let  $\Omega = \mathbb{R}^d \times (0, +\infty)$ ,  $\Gamma_\varepsilon := \Omega \cap (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z})$ ,  $\mathcal{F}$  and  $F_\varepsilon$  satisfy respectively (F1)–(F2) and (S1)–(S3),  $g \in C^{0,1}(\mathbb{R}^d)$  be compactly supported. Assume that  $u : \Omega \rightarrow \mathbb{R}$  and  $v : \Gamma_\varepsilon \rightarrow \mathbb{R}$  are respectively solutions of (4.1.1)–(4.1.2) and (4.1.3). Then there exist an exponent  $\eta \in (0, 1)$  and a constant  $C > 0$  such that*

$$\max_{\Gamma_\varepsilon \cap \{(x,t) \in U \times [0,T]\}} |(u - v)| \leq C\varepsilon^\eta.$$

**Structure of the article:** We start by gathering our notation and some preliminary results in Section 4.1.2. The proof of Theorem 4.1.1 is presented in Section 4.2. Section 4.3 is devoted to the derivation of the maximum principle in the doubled variables.

### 4.1.2 Hypotheses and notation

Throughout this chapter, we work in  $\mathbb{R}^d$  in dimension  $d \geq 2$  and all differential equations and inequalities are to be interpreted in the viscosity sense (voir [34, 45]). The set of real  $d$ -by- $d$  symmetric matrices is denoted by  $\mathbb{S}^d$ . If  $A \in \mathcal{S}_d$ , then  $|A|$  denotes the square root of the largest eigenvalue of  $A^2$ . We write  $A \geq 0$  if  $A$  has nonnegative eigenvalues. The Pucci extremal operators with ellipticity  $0 < \gamma \leq \Gamma$  are defined for each  $A \in \mathcal{S}_d$  by:

$$\mathcal{P}_{\lambda,\Lambda}^+(A) = -\lambda \operatorname{tr}(A_+) + \Lambda \operatorname{tr}(A_-) \quad \text{and} \quad \mathcal{P}_{\lambda,\Lambda}^-(A) = -\Lambda \operatorname{tr}(A_+) + \lambda \operatorname{tr}(A_-).$$

Here  $\operatorname{tr}(A)$  denotes the trace of  $A$ , and  $A_+, A_- \in \mathcal{S}_d$  are the uniquely matrices defined by the conditions:  $A = A_+ - A_-$ ,  $A_+ A_- = 0$  and  $A_+, A_- \geq 0$ . The identity matrix is denoted by  $I_d$ .

The open ball of  $\mathbb{R}^d$  centered at  $x$  of radius  $\rho$  is denoted by  $B_\rho(x)$ . If  $x = 0$ , we simply write  $B_\rho$ . The following elementary cylindrical domains play a central role in the theory: for all  $\rho > 0$  and  $x \in \mathbb{R}^d$ , we define  $Q_\rho(x, t) := B_\rho(x) \times (t - \rho^2, t)$  and denote  $Q_\rho := Q_\rho(0, 0)$ . We also introduce the *forward cylinders*: for all  $\rho > 0$  and  $x \in \mathbb{R}^d$ , we define  $Q_\rho^f(x, t) := B_\rho(x) \times (t, t + \rho^2)$  and denote  $Q_\rho^f := Q_\rho^f(0, 0)$ .

We also use  $|\Omega|$  to denote the Lebesgue measure of the set  $\Omega$ . If  $(x, t), (y, s) \in \Omega$ , we define the parabolic distance between  $(x, t)$  and  $(y, s)$  by

$$\operatorname{dist}_p((x, t), (y, s)) = |x - y| + \sqrt{|t - s|},$$

and we define  $\operatorname{diam} \Omega := \sup\{|x - y| : (x, t), (y, s) \in \Omega\}$ , so  $\operatorname{diam} \Omega$  is not the usual diameter of a set. When  $\Omega$  is a cylinder  $\Omega = U \times (a, b)$ , then  $\operatorname{diam} \Omega$  is the usual cross-section  $U$ , but if  $\Omega$  is noncylindrical,  $\operatorname{diam} \Omega$  may be strictly larger than the diameter of any cross-section  $U(t)$ .

For the modern theory of nonlinear parabolic equations we refer to [92, 104]. A nice introduction to viscosity solutions of nonlinear parabolic equations can be found in [78]. Let  $Q \subseteq U \times (0, T)$  and  $\alpha \in (0, 1]$ . The set of upper and lower semicontinuous functions on  $Q$  are denoted by  $\operatorname{USC}(Q)$  and  $\operatorname{LSC}(Q)$ ,

respectively. We will use the following notation : we say that  $u \in C^{0,\alpha}(Q)$  means that there exists  $C > 0$  such that for all  $(x, t), (y, s) \in Q$ , we have

$$|u(x, t) - u(y, s)| \leq C(|x - y|^\alpha + |t - s|^{\frac{\alpha}{2}}).$$

In other words,  $u$  is  $\frac{\alpha}{2}$ -Hölder continuous in  $t$  and  $\alpha$ -Hölder continuous in  $x$ . Then we recall the definition of the monotone envelope in the parabolic framework (see [78]). If  $\Omega \subseteq \mathbb{R}^{d+1}$  is open, then the *monotone envelope*  $\Gamma_u$  of a lower semi-continuous function  $u : \Omega \rightarrow \mathbb{R}$  is the largest function  $v : \Omega \rightarrow \mathbb{R}$  lying below  $u$  which is non-increasing with respect to  $t$  and convex with respect to  $x$ . The set  $\{u = \Gamma_u\}$  where  $\Gamma(u)$  coincides with  $u$  is called the *contact set*. Moreover, the monotone envelop of a function  $u \in \text{LSC}(\Omega)$  admits the following representation:

$$\Gamma_u(x, t) := \sup_{p \in \mathbb{R}^d} \inf_{(y, s) \in \Omega, s \leq t} (u(y, s) + p \cdot (x - y)). \quad (4.1.4)$$

If  $(x, t) \in \{u = \Gamma_u\}$ , we can define a vector  $(p, h) \in \mathbb{R}^{d+1}$  associated to  $(x, t)$  by

$$h = u(x, t) - p \cdot x = \inf_{(y, s) \in \Omega, s \leq t} \{u(y, s) - p \cdot y\}. \quad (4.1.5)$$

Note also that if  $a > 0$ , then the inclusion  $\{u = \Gamma_u\} \subseteq \{u + at = \Gamma_{u+at}\}$  holds true.

We call a constant *universal* if it depends only on the dimension  $d$  and the ellipticity constants  $\lambda$  and  $\Lambda$ . If  $U \subseteq \mathbb{R}^d$  and  $s < t$ , then the parabolic boundary of  $U \times (s, t)$  is denoted by

$$\partial_p(U \times (s, t)) := U \times \{s\} \cup \partial U \times (s, t).$$

The nonlinear elliptic operator  $\mathcal{F} : \mathbb{S}_d \rightarrow \mathbb{R}$  satisfies:

(F1)  $\mathcal{F}$  is uniformly elliptic and Lipschitz; precisely, we assume that there exist constants  $0 < \lambda \leq 1$  such that, for every  $M, N \in \mathbb{S}_d$ ,

$$\mathcal{P}_{\lambda,1}^-(M - N) \leq \mathcal{F}(M) - \mathcal{F}(N) \leq \mathcal{P}_{\lambda,1}^+(M - N).$$

(F2)  $\mathcal{F}(0) = 0$ .

Let  $\varepsilon > 0$  and  $R_0 \geq 1$ . The lattice is denoted by  $\Gamma_\varepsilon := (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z}) \cap \Omega$  and we define the closure of the mesh by

$$\overline{\Gamma}_\varepsilon := \{(x, t) \in \varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z} : \exists z \in \varepsilon\mathbb{Z}^d \text{ such that } |x - z| \leq R_0\varepsilon \text{ and } (z, t) \in \Gamma_\varepsilon\}.$$

The boundary points  $\partial_p \Gamma_\varepsilon$  of the mesh are given by  $\partial_p \Gamma_\varepsilon := \overline{\Gamma}_\varepsilon \setminus \Gamma_\varepsilon$  and we define the interior lattice  $\text{int } \Gamma_\varepsilon$  by  $\text{int } \Gamma_\varepsilon := \cup \{\Gamma \subseteq \varepsilon\mathbb{Z}^{d+1} : \overline{\Gamma} \subseteq \Gamma_\varepsilon\}$ . If  $U \subseteq \mathbb{R}^d$  and  $s < t$ , we also define  $\Omega_\varepsilon := U_\varepsilon \times [s - \varepsilon^2, t]$  where

$$U_\varepsilon := \{y \in \mathbb{R}^d : |y - z| \leq R_0\varepsilon \text{ for some } z \in U\}$$

The parabolic boundary condition  $g$  satisfies

$$g \in C(\Omega_\varepsilon \setminus \Omega) \quad (4.1.6)$$

This implies in particular that  $\partial_\varepsilon \Lambda \subseteq \text{domain}(g)$  and  $\partial \Gamma_\varepsilon \subseteq \Omega_\varepsilon$ .

For our schemes, we now formulate the assumptions on the discrete operator  $F_\varepsilon$ :

(S1) *Monotonicity*. For every  $\phi, \xi \in \mathcal{E}$  and  $z \in \varepsilon\mathbb{Z}^d$ ,

$$\phi(z) - \xi(z) > \sup_{\varepsilon\mathbb{Z}^d \setminus \{z\}} (\phi - \xi)$$

implies that

$$F_\varepsilon[\phi](z) > F_\varepsilon[\xi](z).$$

(S2) *Consistency.* There exists a constant  $K_0 > 1$  such that, for every  $\phi \in C^3(\mathbb{R}^d)$ ,

$$\sup_{z \in \varepsilon \mathbb{Z}^d} |\mathcal{F}(D^2\phi(z)) - F_\varepsilon[\phi](z)| \leq K_0 \varepsilon \sup_{x \in \mathbb{R}^d} |D^3\phi(x)|.$$

(S3) *Continuity and locality.* There exist constants  $R_0 > 1$  and  $K_1 > 1$  such that, for every  $\xi \in \mathcal{E}$ ,  $\phi \in C^2(\mathbb{R}^d)$  and  $z \in \varepsilon \mathbb{Z}^d$ ,

$$|F_\varepsilon[\xi + \phi](z) - F_\varepsilon[\xi](z)| \leq K_1 \sup_{x \in B_{R_0\varepsilon}(z)} |D^2\phi(x)|.$$

Let us discuss shortly the assumptions above. The assumption (S1) represents the discrete ellipticity of the operator. It can be also seen as a discrete version of the definition of a viscosity subsolution of the elliptic PDE. For instance, the usual centered discretization of the Laplace in 1-D given by

$$F_\varepsilon[\phi](x) = \frac{1}{2\varepsilon^2} \{\phi(x - \varepsilon) - 2\phi(x) + \phi(x + \varepsilon)\},$$

clearly satisfies (S1). The assumption (S3) ensures us that the discrete operator  $F$  is stable by second-order perturbations. In particular, notice that  $F_\varepsilon$  does not see the affine functions. Moreover, the constant  $R_0$  represents the radius of the ball where the scheme picks the point to compute the By the locality assumption (S3), we see that in (S1), we can restrain the sup on the whole grid by a sup taken on the close neighbors of  $z$  which satisfy  $|y - z| \leq R_0\varepsilon$ .

By combining (S1) and (S3) we get

(S1') *Weak monotonicity:* For every  $\phi, \xi \in \mathcal{E}$  and  $z \in \varepsilon \mathbb{Z}^d$ ,

$$\phi(z) - \xi(z) \geq \sup_{\mathbb{Z}^d \setminus \{z\}} (\phi - \xi) \quad \text{implies that} \quad F_\varepsilon[\phi](z) \geq F_\varepsilon[\xi](z).$$

It suffices to consider  $\psi(y) = \phi(y) - a|y - z|^2$  for  $a > 0$ . Observe that  $\psi(z) - \xi(z) > \sup_{\mathbb{Z}^d \setminus \{z\}} (\psi - \xi)$ , applying (S1) gives  $F_\varepsilon[\psi](z) > F_\varepsilon[\xi](z)$ . Then by (S3), we also have

$$|F_\varepsilon[\psi](z) - F_\varepsilon[\phi](z)| \leq aK_1.$$

By sending  $a \rightarrow 0$ , we get (S1').

## 4.2 Proof of the implicit rate of convergence

In this section, we give the proof of Theorem 4.1.1, which is a comparison argument made quantitative by the regularity theory for uniformly parabolic equations. There are two main ingredients: (i) the estimates for quadratic expansions of uniformly parabolic equations found in [51], and (ii) a maximum principle of ABP-type in the doubled variables. The second ingredient is proved in the next section.

A technical difficulty arises because solutions of uniformly parabolic (or elliptic) equations are not in general  $C^2$ . To resolve this issue, we rely on the regularity theory, in particular the parabolic  $W^{2,\sigma}$  and  $W^{3,\sigma}$  estimates (here  $\sigma > 0$  is a small universal constant, see [51]) which give quadratic expansions for solutions of constant-coefficient equations in sets of large measure.

### 4.2.1 The $W^{3,\sigma}$ estimate

In this section we state the parabolic  $W^{3,\sigma}$  estimate for viscosity solutions of the uniformly parabolic equation (4.1.1). Now we define, for  $u : \Omega \rightarrow \mathbb{R}$ , the quantity

$$\Psi(u, \Omega)(x, t) := \inf \left\{ A \geq 0 : \exists (b, \xi, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } \forall (y, s) \in \Omega, s \leq t, \right. \\ \left. |u(y, s) - u(x, t) - b(s - t) - \xi \cdot (y - x) - \frac{1}{2}(y - x) \cdot M(y - x)| \leq \frac{1}{6}A \left( |x - y|^3 + |s - t|^{3/2} \right) \right\}.$$

We emphasize here that  $\Psi(u, \Omega)(x, t)$  is defined in terms of cubic polynomials that touch  $u$  at  $(x, t)$  and stay below or above  $u$  in the whole domain  $\Omega$  restricted to times less than  $t$ . The statement of the parabolic  $W^{3,\sigma}$  estimate is given by the following theorem.

**Proposition 4.2.1** (Parabolic  $W^{3,\sigma}$  estimate). *Assume  $F$  satisfies (F1)-(F3), and  $u \in C(Q_1)$  solves (4.1.1) in  $Q_1$ . Then there exist universal constants  $C, \sigma > 0$  such that, for all  $\kappa > 0$ ,*

$$|\{(x, t) \in Q_{1/2}(0, -\tfrac{1}{4}) : \Psi(u, Q_{3/4})(x, t) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{Q_1} |u|} \right)^{-\sigma}.$$

## 4.2.2 A comparison principle for implicit schemes

**Proposition 4.2.2.** *Fix  $\varepsilon > 0$ . Suppose that  $F_\varepsilon$  satisfies (S1)-(S3). Let  $Q = (U \times [S, T])$  with  $U \subseteq \mathbb{R}^d$  and  $\hat{Q} := Q \cap \Gamma_\varepsilon$  such that  $\hat{Q} \neq \emptyset$ . Assume that  $u, v : \hat{Q} \rightarrow \mathbb{R}$  satisfy*

$$\hat{\partial}_t^\varepsilon u + F_\varepsilon[u] \leq \hat{\partial}_t^\varepsilon v + F_\varepsilon[v] \quad \text{on } \hat{Q}.$$

*Then, for all time  $t \in [S, T]$ , we have*

$$\max_{\hat{Q} \cap \{(z,s): s \leq t\}} (u - v) \leq \max_{\partial_p \hat{Q} \cap \{(z,s): s \leq t\}} (u - v).$$

*Proof.* It is clear that the result holds at the initial time  $\min\{s : \hat{Q} \cap (\mathbb{R} \times \{s\}) \neq \emptyset\}$ . Arguing by contradiction, consider the first time  $t_0 > 0$  such that there exists  $(z_0, t_0) \in \hat{Q}$  such that

$$(u - v)(z_0, t_0) = \max_{\hat{Q} \cap \{(z,s): s \leq t_0\}} (u - v) > \max_{\partial_p \hat{Q} \cap \{(z,s): s \leq t_0\}} (u - v).$$

First we can select  $\nu > 0$  such that

$$(u - v)(z_0, t_0) \geq \nu + \max_{\partial_p \hat{Q} \cap \{(z,s): s \leq t_0\}} (u - v). \quad (4.2.1)$$

Then we introduce the function  $\tilde{u}$  given by

$$\tilde{u}(z, s) := u(z, s) - \frac{1}{2}\theta|z - z_0|^2 \quad \text{for } s \in \{t_0 - \varepsilon^2, t_0\},$$

where  $\theta > 0$  is sufficiently small and will be selected below. Since the map  $z \mapsto (\tilde{u} - v)(z, t_0)$  has a strict global maximum at  $z_0$ , we deduce from (S1) that

$$F_\varepsilon[v(\cdot, t_0)](z_0) < F_\varepsilon[\tilde{u}(\cdot, t_0)](z_0).$$

Moreover, by using (S3), we have

$$|F_\varepsilon[\tilde{u}(\cdot, t_0)](z_0) - F_\varepsilon[u(\cdot, t_0)](z_0)| \leq K_1\theta$$

and this implies that

$$F_\varepsilon[v(\cdot, t_0)](z_0) \leq F_\varepsilon[u(\cdot, t_0)](z_0) + K_1\theta.$$

By the assumption of the proposition, we have in particular at  $(z_0, t_0)$ ,

$$\begin{aligned} u(z_0, t_0) - u(z_0, t_0 - \varepsilon^2) &\leq v(z_0, t_0) - v(z_0, t_0 - \varepsilon^2) + F_\varepsilon[v(\cdot, t_0)](z_0) - F_\varepsilon[u(\cdot, t_0)](z_0) \\ &\leq v(z_0, t_0) - v(z_0, t_0 - \varepsilon^2) + K_1\theta. \end{aligned}$$

After rearranging the terms and choosing  $\theta = \nu/(2K_1)$ , we get

$$\begin{aligned} u(z_0, t_0) - v(z_0, t_0) &\leq u(z_0, t_0 - \varepsilon^2) - v(z_0, t_0 - \varepsilon^2) \leq \max_{\partial_p \hat{Q} \cap \{(y,s): s \leq t_0 - \varepsilon^2\}} (u - v) + K_1\theta \\ &\leq \max_{\partial_p \hat{Q} \cap \{(y,s): s \leq t_0\}} (u - v) + \frac{\nu}{2}. \end{aligned}$$

This yields a contradiction with (4.2.1) and completes the proof.  $\square$

### 4.2.3 A maximum principle in the doubled variables

The following proposition is a double-variable comparison principle formulated with the proof of Theorem 4.1.1 in mind. Both its statement and proof are analogous to that of [4, Lemma 6.1]. It gives a lower bound for the Lebesgue measure in  $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$  of the set of points at which we can touch the difference of a sub solution  $u$  and super solution  $v$  by planes, after doubling the variables and adding the usual quadratic penalization term. Its proof is postponed to Section 4.3.

**Proposition 4.2.3.** *Let  $\Omega \subseteq \mathbb{R}^{d+1}$  and set  $\Gamma_\varepsilon := \Omega \cap (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z})$ . Assume that  $u \in C^{0,\alpha}(\overline{\Omega})$  and  $v : \Gamma_\varepsilon \rightarrow \mathbb{R}$  satisfy*

$$\partial_t u + \mathcal{P}_{1,\Lambda}^-(D^2 u) \leq 1 \quad \text{in } \Omega \quad (4.2.2)$$

and

$$\hat{\partial}_t^\varepsilon v + F_\varepsilon[v] \geq -1 \quad \text{in } \Gamma_\varepsilon.$$

Fix  $0 < \delta \leq 1$  and define, for each  $(p, q, h, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ , a function  $\Phi : \Omega \times \Gamma_\varepsilon \rightarrow \mathbb{R}$  by

$$\Phi_{p,q,h,k}(x, t, y, s) := u(x, t) - v(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s - h)^2) - p \cdot x - q \cdot y - k.$$

Assume that  $V \subseteq \mathbb{R}^{d+1} \times (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z})$  and  $W \subseteq \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$  are such that  $\overline{V} \subseteq \Omega \times \Gamma_\varepsilon$  and, for each  $(p, q, h, k) \in W$ , there exists  $\hat{k} \in \mathbb{R}$  and  $(x, t, y, s) \in V$  such that

$$0 = \Phi_{p,q,h,\hat{k}}(x, t, y, s) = \sup \{ \Phi_{p,q,h,\hat{k}}(\xi, \tau, \eta, \sigma) : (\xi, \tau, \eta, \sigma) \in \Omega \times \Gamma_\varepsilon, \tau \leq t, \sigma \leq s \}$$

and

$$|k - \hat{k}| \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \delta^{-1/2} \varepsilon^2.$$

Then there exists a constant  $C > 1$  depending only on  $d, \Lambda, \text{diam } \Omega$  and  $\|u\|_{C^{0,\alpha}(\overline{\Omega})}$  such that for all  $\rho > 0$ , we have

$$|W| \leq C \delta^{-2(d+1)} \left( 1 + \frac{\varepsilon^\alpha + \delta^{1/2}}{\rho} \right)^{2d+2} |V + Q_\rho^f \times Q_\rho^f|$$

with  $Q_\rho^f = B_\rho \times (0, \rho^2)$ .

### 4.2.4 Proof of the main theorem

We now show that Theorem 4.1.1 is a consequence of Propositions 4.2.1, 4.2.3 and 4.5.1. The proof of Proposition 4.2.5 is presented in Section 4.5. The proof of Proposition 4.2.3 is presented in Section 4.3.

We require the initial data  $g$  to be continuous, not identically zero with compact support. In particular,  $g$  satisfies the condition

$$0 \leq g(x) \leq C_0 e^{-B|x|^2} \quad (4.2.3)$$

for some constants  $B, C_0 > 0$ . Other boundary conditions and parabolic boundaries could be considered. Now we give a lemma which will be useful to control both the behavior of the viscosity solution and the approximate solution far from the origin in space or time by using some properties established by Armstrong and Trokhimtchouk [7].

**Lemma 4.2.4** (Lemma 4.1 and Corollary 3.11, [7]). *For each  $\tau > \frac{1}{4B\Lambda}$ , there exists a constant  $C > 0$ , depending only on  $C_0$  and  $\tau$ , such that for all  $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$ ,*

$$u(x, t) \leq C \Phi(x, t + \tau).$$

Moreover, for each  $0 < a_1 < (4\Lambda)^{-1} \leq (4\lambda)^{-1} < a_2$ , there exists a constant  $C > 1$  such that, for all  $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$ , we have

$$\Phi^+(x, t) \leq C t^{-\alpha^+} \exp(-a_1 |x|^2 / t) \quad (4.2.4)$$



First we give the rate of convergence of the parabolic schemes near the boundary.

**Proposition 4.2.5** (Rate far the origin in space and for small times). *There exists constants  $R_1$  and  $T_1$  depending on  $B, C_0$  given by (4.2.3) and the ellipticity constants  $\lambda, \Lambda$  and the dimension  $d$  such that*

$$\max\{|u - v|(x, t) : |x| \geq R_1 \text{ or } t \geq T_1\} \leq \varepsilon^2. \quad (4.2.5)$$

Moreover there exists a universal constant such that for all  $(x, t) \in (\mathbb{R}^d \times [0, T]) \cap \Gamma_\varepsilon$ , we have

$$|u - v|(x, t) \leq C(1 + \|g\|_{C^{0,1}})(\varepsilon + \text{dist}((x, t), \Gamma_\varepsilon)). \quad (4.2.6)$$

The next proposition presents the main result about the rate of convergence inside the cylinder  $B_{R_1} \times [0, T]$ .

**Proposition 4.2.6.** *Let  $\Omega = B_{R_1} \times [0, T] \subseteq \mathbb{R}^{d+1}$  and  $\Gamma_\varepsilon := \Omega \cap (\varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z})$ . Let  $\mathcal{F}$  and  $F_\varepsilon$  satisfy respectively (F1) and (S1)–(S3),  $g \in C^{0,1}(\mathbb{R}^d)$ . Assume that  $u : \Omega \rightarrow \mathbb{R}$  and  $v : \Gamma_\varepsilon \rightarrow \mathbb{R}$  are respectively solutions of (4.1.1)–(4.1.2) and (4.1.3). Then there exist an exponent  $\eta \in (0, 1)$  and a constant  $C > 0$  such that*

$$\max_{\Gamma_\varepsilon \cap \{(x, t) \in B_{R_1} \times [0, T]\}} |(u - v)| \leq C\varepsilon^\eta.$$

**Proof of Proposition 4.2.6.** The proof of the rate is organized as follows. Arguing by contradiction, we assume the error is too large compared to the proposed rate. We study the monotone maximum points associated to the family of double-variable functions defined in Proposition 4.2.3. This family of functions is indexed by a set of parameters  $(p, q, h, k) \in \mathbb{R}^{2d+2}$  taking values in some ranges whose lengths depends on the error. Steps 2-3 guarantee that we can satisfy the conditions of Proposition 4.2.3. By applying this double-variable proposition, we exhibit a set of points realizing the supremum of the functions in a monotone way. Then, in Steps 4-5, if the error is too large, we show we can pick among these points a point admitting a quadratic expansion with a controlled cubic term in space and time thanks to the parabolic  $W^{3,\sigma}$  estimate given by Proposition 4.2.1. Finally we get a contradiction in Step 6.

For notational simplicity, we introduce the cumulated error  $E(t)$  for times  $s$  less than  $t$  given by

$$E(t) := \max_{(y, s) \in \Gamma_\varepsilon, s \leq t} |u(y, s) - v(y, s)|.$$

At the initial time  $t = 0$ , it is clear that  $E(0) = 0$ . Arguing by contradiction, assume that there exists  $(z_0, T_0) \in \Gamma_\varepsilon$ ,  $T_0 > 0$  such that,

$$(u - v)(z_0, T_0) = E(T_0) = \max_{(y, t) \in \Lambda, t \leq T_0} (u - v)(y, t) > c\varepsilon^\eta.$$

with  $\eta > 0$  to be selected below.

*Step 1.* We make several preliminary observations. First we may assume without loss of generality that  $\Omega \subseteq Q_1$  and  $K_0 = 1$  by using some rescaling and normalizing arguments. Moreover, up to replace  $u$  by  $\tilde{u}(x, t) = u(x, t) - \frac{1}{2}E^\beta t$ , we can assume that

$$\partial_t u + \mathcal{F}(D^2 u) = -\frac{1}{2}E^\beta,$$

with  $\beta > 0$  to be selected below.

In view of the bound  $K_0 \leq 1$  and the smoothness of  $\Omega$ , the global Hölder estimates yield, for  $\alpha(d, \Gamma) \in (0, 1)$ ,

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C.$$

Since  $u = g$  on  $\partial_p \Omega$  and  $v = g$  on  $\partial_p \Gamma_\varepsilon$ , we show that for every  $(x, t, z, s) \in \Omega \times \Gamma_\varepsilon$ , we have

$$|u(x, t) - v(z, s)| \leq C \text{dist}_p((x, t), (z, s))^\alpha + C \text{dist}_p((z, s), \partial_p \Lambda)^\alpha + C(1 + |g|_{C^{0,1}})\varepsilon. \quad (4.2.7)$$

To get this inequality, we write the following decomposition

$$|u(x, t) - v(z, s)| \leq |u(x, t) - u(z, s)| + |u(z, s) - u(z_\Omega, s_\Omega)| + |u(z_\Omega, s_\Omega) - v(z_1, s_1)| + |v(z_1, s_1) - v(z, s)|,$$

where  $(z_\Omega, s_\Omega) \in \partial_p \Omega$  and  $(z_1, s_1) \in \partial_p \Gamma_\varepsilon$  satisfy

$$\text{dist}_p((z, s), (z_\Omega, s_\Omega)) = \text{dist}_p((z, s), \partial_p \Omega)$$

and

$$\text{dist}_p(z_\Omega, s_\Omega, (z_1, s_1)) = \text{dist}_p((z_\Omega, s_\Omega), \partial_p \Gamma_\varepsilon).$$

By using the  $C^{0,\alpha}$  estimates on the viscosity solution  $u$  up to the boundary, we know that

$$|u(x, t) - u(z, s)| \leq C \text{dist}_p((x, t), (z, s))^\alpha$$

and

$$|u(z, s) - u(z_\Omega, s_\Omega)| \leq C \text{dist}_p((z, s), \partial_p \Omega)^\alpha.$$

Then by using that  $g \in C^{0,1}$  and noticing that  $\text{dist}_p(z_\Omega, s_\Omega, (z_1, s_1)) \leq (1 + \sqrt{d})\varepsilon$ , we deduce that

$$\begin{aligned} |u(z_\Omega, s_\Omega) - v(z_1, s_1)| &= |g(z_\Omega, s_\Omega) - g(z_1, s_1)| \\ &\leq |g|_{C^{0,1}} \left( |z_\Omega - z_1| + |s_\Omega - s_1|^{1/2} \right) \leq (1 + \sqrt{d}) |g|_{C^{0,1}} \varepsilon. \end{aligned}$$

Finally Proposition 4.5.1 yields that

$$\begin{aligned} |v(z_1, s_1) - v(z, s)| &= |g(z_1, s_1) - v(z, s)| \\ &\leq C(1 + \|g\|_{C^{0,1}})(\varepsilon + \text{dist}((z, s), (z_1, s_1))) \\ &\leq C(1 + \|g\|_{C^{0,1}})(\varepsilon + \text{dist}((z, s), \partial_p \Gamma_\varepsilon)), \end{aligned}$$

since  $\text{dist}((z, s), (z_1, s_1)) \leq \text{dist}((z, s), \partial_p \Gamma_\varepsilon) + \varepsilon$ . For convenience we may take  $0 < \sigma \leq 1/2$ .

*Step 2.* We define, for each  $(p, q, h, k) \in \mathbb{R}^{2d+2}$ , the auxiliary function  $\Phi : \Omega \times \Gamma_\varepsilon \rightarrow \mathbb{R}$  by

$$\Phi_{p,q,h,k}(x, t, z, s) := u(x, t) - v(z, s) - \frac{1}{2\delta} (|x - z|^2 + (t - s - h)^2) - p \cdot x - q \cdot z - k,$$

for some  $\delta > 0$  to be determined. Choose  $(z_0, T_0) \in \Gamma_\varepsilon$  such that  $\Phi_{0,0,0,0}(z_0, T_0, z_0, T_0) = E(T_0)$ . We compute

$$\begin{aligned} \Phi_{p,q,h,\hat{k}}(z_0, T_0, z_0, T_0) &= E(T_0) - (p + q) \cdot z_0 - \hat{k} \geq E(T_0) - (|p| + |q|)|z_0| - \hat{k} \\ &\geq E(T_0) - 2r(\text{diam } \Omega + 1) - \hat{k} \geq \frac{7}{8}E(T_0) - \hat{k}. \end{aligned}$$

Since  $\varepsilon \leq 1/2$ , it is clear that  $\text{diam } \bar{\Gamma}_\varepsilon \leq \text{diam } \Omega + 1$ . Set  $r := \min\{\frac{1}{8} \frac{1}{\text{diam } \Omega + 1} E(T_0), 1\}$ . Given  $p, q \in B_r$  and  $0 \leq h \leq r$  and  $k \in [7/8E - r, 7/8E]$ .

Let  $0 \leq s, t \leq T_0$ . By definition of  $\Phi$  and using Step 1, we have

$$\begin{aligned} \Phi_{p,q,h,\hat{k}}(x, t, z, s) &= u(x, t) - v(z, s) - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) - p \cdot x - q \cdot z - \hat{k} \\ &\leq C \text{dist}((z, s), \partial_p \Gamma_\varepsilon)^\sigma + C|x - z|^\sigma + C|t - s|^{\sigma/2} + C(1 + |g|_{C^{0,1}})\varepsilon \\ &\quad - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) + r(2 \text{diam } \Omega + 1) - \hat{k}. \end{aligned}$$

Then we obtain by using that  $(\text{diam } \Omega + 1)r \leq \frac{1}{4}E(T_0)$  that

$$\begin{aligned} \Phi_{p,q,h,\hat{k}}(x, t, z, s) &\leq C \text{dist}((z, s), \partial_p \Gamma_\varepsilon)^\sigma + C(1 + |g|_{C^{0,1}}) \varepsilon + \frac{1}{4}E(T_0) \\ &\quad + \left( CE^{-(4-\sigma)/\sigma}(T_0) - \frac{1}{2\delta} \right) (|x - z|^2 + (t - s)^2) + \frac{1}{4}E(T_0) - \hat{k} \\ &\leq C \text{dist}((z, s), \partial_p \Gamma_\varepsilon)^\sigma + \frac{1}{2}E(T_0) - \hat{k}. \end{aligned}$$

In the first line, we get by Young's inequality that

$$|x - z|^\sigma = E^{(2-\sigma)/2}(T_0)(E^{-(2-\sigma)/2}(T_0)|x - z|^2)^{\sigma/2} \leq \frac{1}{8}E(T_0) + CE^{-(2-\sigma)/\sigma}(T_0)|x - z|^2$$

and

$$|t - s|^{\sigma/2} = E^{(2-\sigma/2)/2}(T_0)(E^{-(2-\sigma/2)/2}(T_0)(t - s)^2)^{\sigma/4} \leq \frac{1}{8}E(T_0) + CE^{-(4-\sigma)/\sigma}(T_0)(t - s)^2.$$

To get the second inequality, we must impose the condition  $\delta \leq cE^{(4-\sigma)/\sigma}(T_0)$ . Then we may fix

$$\delta = cE^{(4-\sigma)/\sigma}(T_0). \quad (4.2.8)$$

so that, for all  $(p, q, h, k) \in B_r \times B_r \times [0, r] \times [7/8E - r, 7/8E]$ , the map

$$(x, t, z, s) \mapsto \Phi_{p,q,h,k}(x, t, z, s)$$

attains its supremum in a monotone way in time in  $\Omega \times \Gamma_\varepsilon$  on  $\overline{\Omega} \times (\Gamma_\varepsilon)_a$  where  $a := cE^{1/\sigma}(T_0)$ . Actually, since  $|x - z|^2 + (t - s)^2 \leq \delta$  is a monotone maximum point, up to modify  $a$ , the supremum is attained in a monotone way in time in  $\Omega \times \Lambda$  on  $\overline{\Omega}_a \times (\Gamma_\varepsilon)_a$  where  $a := cE^{1/\sigma}(T_0)$ . Here we have denoted

$$\overline{\Omega}_a := \{(x, t) \in \Omega : \text{dist}((x, t), \partial_p \Omega) > a\}$$

and

$$(\Gamma_\varepsilon)_a := \{(x, t) \in \Gamma_\varepsilon : \text{dist}((x, t), \partial_p \Gamma_\varepsilon) > a\}.$$

*Step 3.* We show that for all  $(p, q, h, k) \in B_r \times B_r \times (0, r) \times [7/8E - r, 7/8E]$ , there exists  $\hat{k} \in \mathbb{R}$  satisfying

$$|\hat{k} - k| \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \frac{\varepsilon^2}{\delta^{1/2}}$$

and the map

$$(x, t, z, s) \mapsto \Phi_{p,q,h,\hat{k}}(x, t, z, s)$$

attains its supremum in a monotone way in time in  $\Omega \times \Gamma_\varepsilon$  on  $\overline{\Omega}_a \times (\Gamma_\varepsilon)_a$  where  $a := cE^{1/\sigma}(T_0)$  and this supremum is equal to 0.

We are going to reduce the proof of the claim to a suitable inequality. For all  $(p, q, h) \in B_r \times B_r \times (0, r)$ , we set

$$\Psi_{p,q,h}(x, t, z, s) := u(x, t) - v(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s - h)^2) - p \cdot x - q \cdot y$$

and

$$k(t) := \sup_{\substack{(x,\tau,y,s) \in \Omega \times \Gamma_\varepsilon \\ \tau \leq t}} \Psi_{p,q,h}(x, \tau, z, s).$$

Assume that we have shown that for all  $0 \leq t \leq T - \varepsilon^2$ , we have

$$k(t + \varepsilon^2) \leq \max \left\{ C(1 + \|g\|_{C^{0,1}}) \max \left\{ \varepsilon, \delta^{1/4} \right\} + 2r \text{diam } \Omega, k(t) + \|u\|_{C^{0,\alpha}} \varepsilon^\alpha + \frac{\varepsilon^2}{\delta^{1/2}} \right\}. \quad (4.2.9)$$

Then, for each  $k \in [7/8E - r, 7/8E]$ , the first alternative in (4.2.9) cannot happen by using  $k(T_0) \geq 7/8E(T_0)$  and  $\delta^{1/4} \ll E$  (by (4.2.8)). Hence we deduce by monotonicity of  $k(\cdot)$  that for all  $k \in [7/8E - r, 7/8E]$ ,

$$0 \leq k(t + \varepsilon^2) - k(t) \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \frac{\varepsilon^2}{\delta^{1/2}},$$

which implies the claim given by Step 3 by invoking Step 2.

It remains to prove (4.2.9). It is clear that  $k(t + \varepsilon^2) = \max\{S_1, S_2\}$  where we set

$$S_1 := \sup_{\substack{(x,\tau,y,s) \in \Omega \times \Gamma_\varepsilon \\ \tau=0}} \Psi_{p,q,h}(x, \tau, z, s) \quad \text{and} \quad S_2 := \sup_{\substack{(x,\tau,y,s) \in \Omega \times \Gamma_\varepsilon \\ \varepsilon^2 \leq \tau \leq t + \varepsilon^2}} \Psi_{p,q,h}(x, \tau, z, s).$$

To complete the proof, we are going to show that

$$S_1 \leq C(1 + \|g\|_{C^{0,1}}) \max\{\varepsilon, \delta^{1/4}\} + 2r(\text{diam } \Omega) \quad \text{and} \quad S_2 \leq k(t) + \|u\|_{C^{0,\alpha}(\overline{\Omega})} \tau^\alpha + \frac{\varepsilon^2}{\delta^{1/2}}.$$

By Step 1, we deduce that

$$S_1 \leq \sup \left\{ |u(x, 0) - v(y, s)| : (x, 0, y, s) \in \Omega \times \Gamma_\varepsilon, \max\{|x - y|, |s|\} \leq C\delta^{1/2} \right\} + (|p| + |q|) \text{diam } \Omega.$$

If the sup is attained for  $s = 0$  then  $S_1 \leq C \|g\|_{C^{0,1}} \delta^{1/2}$  by using that  $g$  is Lipschitz. Otherwise, fix  $x_1 \in \mathbb{R}^d$ . Let  $x_0 \in \varepsilon \mathbb{Z}^d$  such that  $|x_1 - x_0| < \varepsilon$ . Then

$$\begin{aligned} |u(x_1, 0) - v(y, s)| &\leq |u(x_1, 0) - u(x_0, 0)| + |u(x_0, 0) - v(y, s)| \\ &\leq |g(x_1) - g(x_0)| + |g(x_0) - v(y, s)| \\ &\leq \|g\|_{C^{0,1}} |x_1 - x_0| + \|g\|_{C^{0,1}} R_0 \varepsilon + C \text{dist}_p((x_0, 0)(y, s)) \\ &\leq \|g\|_{C^{0,1}} (1 + R_0) \varepsilon + C(1 + \|g\|_{C^{0,1}}) \delta^{1/4}, \end{aligned}$$

which implies the desired estimate on the first sup. In the second sup in the last line, we get by a change of variable  $\tau = \tau' + \varepsilon^2$  that

$$S_2 = \sup_{(x,\tau) \in \Omega, 0 \leq \tau \leq t} \Psi_{p,q,h}(x, \tau + \varepsilon^2, z, s).$$

By using that  $u \in C^{0,\alpha}(\overline{\Omega})$ , we deduce that

$$\begin{aligned} \Psi_{p,q,h}(x, \tau + \varepsilon^2, z, s) &\leq u(x, \tau) + \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha - v(y, s) \\ &\quad - \frac{1}{2\delta} (|x - y|^2 + (\tau - s - h)^2 + \varepsilon^4 + 2\varepsilon^2(\tau - s - h)) - p \cdot x - q \cdot y. \end{aligned}$$

By using Step 1, we know that  $|t - s - h| \leq C\delta^{1/2}$  which, by passing to the sup, implies the proposed upper bound on  $S_2$ .

*Step 4.* Let  $Z$  be the set of points where such supremums in a monotone way in time are attained:

$$\begin{aligned} Z &:= \left\{ (x, t, z, s) \in \Omega_a \times (\Gamma_\varepsilon)_a : \exists (p, q, h, k) \in B_r \times B_r \times [0, r] \times [7/8E - r, 7/8E] \right. \\ &\quad \left. \text{s.t. } \exists \hat{k} \in \mathbb{R} \text{ satisfying } |\hat{k} - k| \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \frac{\varepsilon^2}{\delta^{1/2}} \text{ and} \right. \\ &\quad \left. 0 = \Phi_{p,q,h,\hat{k}}(x, t, z, s) = \sup_{\substack{(\xi, \tau, \eta, \sigma) \in \Omega_a \times \Lambda_a \\ \tau \leq t}} \Phi_{p,q,h,\hat{k}}(\xi, \tau, \eta, \sigma) \right\}, \end{aligned}$$

and for  $\rho > 0$ , the inflated set  $\tilde{Z}_\rho$  of  $Z$  denotes the set given by

$$\tilde{Z}_\rho := \left\{ (\tilde{x}, \tilde{t}, \tilde{z}, \tilde{s}) \in (\Omega_a + Q_\rho^f) \times ((\Gamma_\varepsilon)_a + Q_\rho^f) : (\tilde{x}, \tilde{t}) \in Q_\rho^f(x, t), (\tilde{z}, \tilde{s}) \in Q_\rho^f(z, s) \text{ and } (x, t, z, s) \in Z \right\}.$$

where  $Q_\rho^f(x, t)$  is the forward centered cube at  $(x, t)$  (see Section 4.1.2 for notation). Applying Proposition 4.2.3 yields a lower bound on  $|\tilde{Z}_\rho|$  given by

$$|\tilde{Z}_\rho| \geq c\delta^{2d+2} \left(1 + \frac{\varepsilon^\alpha + \delta^{1/2}}{\rho}\right)^{-2d-2} r^{2d+2},$$

Let  $\pi_1 : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$  be the projection onto the first  $d+1$  variables, i.e.,  $\pi_1(x, z) := x$ . Observe that

$$|\pi_1(\tilde{Z}_\rho)| \geq |(\widetilde{\Gamma_\varepsilon})_a^\rho|^{-1} |\tilde{Z}_\rho|.$$

where  $(\widetilde{\Gamma_\varepsilon})_a^\rho := Q_\rho^f + (\Gamma_\varepsilon)_a$  is the inflated set from  $(\Gamma_\varepsilon)_a$ . For an upper bound on  $|(\widetilde{\Gamma_\varepsilon})_a^\rho|$ , the inclusion  $(\Gamma_\varepsilon)_a \subseteq \Gamma_\varepsilon$  gives in particular  $(\widetilde{\Gamma_\varepsilon})_a^\rho \subseteq Q_\rho^f + Q_1$  which implies

$$|(\widetilde{\Gamma_\varepsilon})_a^\rho| \leq (1 + \rho)^d (1 + \rho^2).$$

For the particular choice  $\rho = \varepsilon^\alpha + \delta^{1/2} \leq C$ , we get

$$|\tilde{Z}_\rho| \geq c\delta^{2d+2} r^{2d+2} \geq c(E^{(4-\sigma)/\sigma}(T_0))^{2d+2} E^{2d+2}(T_0) = cE^{(8d+8)/\sigma}(T_0).$$

and

$$|(\widetilde{\Gamma_\varepsilon})_a^\rho| \leq C.$$

Then, by combining the two inequalities above, we obtain

$$|\pi_1(\tilde{Z}_\rho)| \geq |(\widetilde{\Gamma_\varepsilon})_a^\rho|^{-1} |\tilde{Z}_\rho| \geq c|\tilde{Z}_\rho| \geq cE^{(8d+8)/\sigma}(T_0), \quad (4.2.10)$$

Finally, we show that for every  $(x, t, z, s) \in Z$ , we can see from  $\Phi_{p,q,h,k}(x, t, z, s) \geq 0$  for some  $p, q \in B_1$  and  $\sigma < 1/2$  that

$$|x - z|^2 + (t - s)^2 \leq C\delta \leq CE^{(4-\sigma)/\sigma}(T_0) \leq CE^3(T_0). \quad (4.2.11)$$

*Step 4.* We show that there are points  $(x, t, z, s) \in \tilde{Z}_\rho$  such that  $u$  has an appropriate quadratic expansion at  $(x, t)$ . Let  $P_\kappa$  be the set of points at which  $u$  has a quadratic expansion with both a quadratic term of size  $\kappa > 0$  and a cubic error term of size  $\kappa > 0$ :

$$P_\kappa := \left\{ (x, t) \in \Omega : \exists (b, \xi, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } |A| \leq \kappa \text{ and, for all } (y, s) \in \Omega, s \leq t, \right. \\ \left. |u(y, s) - u(x, t) - b(s - t) - \xi \cdot (y - x) - \frac{1}{2}(y - x) \cdot A(y - x)| \leq \frac{1}{6}\kappa \left( |x - y|^3 + |s - t|^{3/2} \right) \right\}.$$

According to the  $W^{3,\sigma}$  estimate (see Proposition 4.2.1), we have

$$|\Omega \setminus P_\kappa| \leq C \left( \frac{\kappa}{1 + \frac{K}{2}\varepsilon^\eta} \right)^{-\sigma} \leq C\kappa^{-\sigma},$$

where the exponent  $\sigma > 0$  depends only on  $d$  and  $\Lambda$  (we can reuse the symbol  $\sigma$  by taking the minimum of this  $\sigma$  with the one from Step 1). In view of (4.2.10), for every  $\kappa \geq CE^{-(8d+8)/\sigma^2}(T_0)$ ,

$$|\Omega \setminus P_\kappa| < |\pi_1(\tilde{Z}_\rho)|.$$

Therefore, we take

$$\kappa = CE^{-(8d+8)/\sigma^2} \quad (4.2.12)$$

to be a fixed constant, which will be selected below. In particular,  $\pi_1(\tilde{Z}_\rho) \cap P_\kappa \neq \emptyset$ .

*Step 5.* By the previous step, there exists  $(x_0, t_0, z_0, s_0) \in \tilde{Z}_\rho$  with  $(x_0, t_0) \in P_\kappa$ . In particular, we deduce there exists  $(x_1, t_1, z_1, s_1) \in Z$  such that  $(x_0, t_0) \in Q_\rho^f(x_1, t_1)$  and  $(z_0, s_0) \in Q_\rho^f(z_1, s_1)$ . Select  $p, q \in B_r$ ,  $h \in [0, r]$  and  $k \in [7/8E - r, 7/8E]$  such that

$$\Phi_{p,q,h,\hat{k}}(x_1, t_1, z_1, s_1) = \sup_{\substack{(x,t,z,s) \in \Omega \times \Gamma_\varepsilon \\ t \leq t_1}} \Phi_{p,q,h,\hat{k}}(x, t, z, s) \quad (4.2.13)$$

for a certain  $\hat{k} \in \mathbb{R}$  such that  $|\hat{k} - k| \leq \|u\|_{C^{0,\alpha}} \varepsilon^\alpha + \frac{\varepsilon^2}{\delta^{1/2}}$ . and  $(b, \xi, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}_d$  such that  $|A| \leq \kappa$  and, for all  $(x, t) \in \Omega$ ,  $t \leq t_0$ , we have

$$\begin{aligned} |u(x, t) - u(x_0, t_0) - b(t - t_0) - \xi \cdot (x - x_0) - \frac{1}{2}(x - x_0) \cdot A(x - x_0)| \\ \leq \frac{1}{6}\kappa \left( |x - x_0|^3 + |t - t_0|^{3/2} \right). \end{aligned} \quad (4.2.14)$$

Up to replace  $A$  by  $(A + A^\top)/2$ , we can assume that  $A \in \mathbb{S}_d$ . Note that  $b + \mathcal{F}(A) = -\frac{1}{2}E^\beta$ , since  $u$  satisfies  $\partial_t u + \mathcal{F}(D^2 u) = -\frac{1}{2}E^\beta$  in  $\Omega$  and  $u$  is touched from above and below at  $(x_0, t_0)$  by cubic polynomials with Hessians equal to  $A$  at  $(x_0, t_0)$ . Combining (4.2.13) and (4.2.14) gives

$$\begin{aligned} u(x_1, t_1) - v(z_1, s_1) - \frac{1}{2\delta} (|x_1 - z_1|^2 + (t_1 - s_1)^2) \\ \geq \sup_{\substack{(x, t, z, s) \in \Omega \times \Gamma_\varepsilon \\ t \leq t_1}} \left( \phi(x, t) - v(z, s) - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) - q \cdot (z - z_1) \right), \end{aligned} \quad (4.2.15)$$

where  $\phi$  is the cubic polynomial defined by

$$\begin{aligned} \phi(x, t) := u(x_0, t_0) + b(t - t_0) + \xi \cdot (x - x_0) - p \cdot (x - x_1) + \frac{1}{2}(x - x_0) \cdot A(x - x_0) \\ - \frac{1}{6}\kappa \left( |x - x_0|^3 + |t - t_0|^{3/2} \right). \end{aligned}$$

In particular, this implies

$$\begin{aligned} u(x_1, t_1) - v(z_1, s_1) - \frac{1}{2\delta} (|x_1 - z_1|^2 + (t_1 - s_1)^2) \\ \geq \sup_{\substack{(x, t, z, s) \in \Omega \times \Gamma_\varepsilon \\ t \leq t_1, s \leq s_1}} \left( \phi(x, t) - v(z, s) - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) - q \cdot (z - z_1) \right), \end{aligned} \quad (4.2.16)$$

Observe that for each  $(z, s) \in \Gamma_\varepsilon$ ,  $s \leq s_1$ , the choice  $x = x_1 + z - z_1$ ,  $t = t_1 + s - s_1 \leq t_1$  yields

$$\begin{aligned} \sup_{(x, t) \in \Omega, t \leq t_1} \left( \phi(x, t) - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) \right) \\ \geq \phi(x_1 + z - z_1, t_1 + s - s_1) - \frac{1}{2\delta} (|x_1 - z_1|^2 + (t_1 - s_1)^2) \\ \geq u(x_0, t_0) + b(t_1 - t_0 + s - s_1) + \xi \cdot (x_1 - x_0 + z - z_1) - p \cdot (z - z_1) \\ + \frac{1}{2}(x_1 - x_0 + z - z_1) \cdot A(x_1 - x_0 + z - z_1) \\ - \frac{1}{6}\kappa \left( |x_1 - x_0 + z - z_1|^3 + |t_1 - t_0 + s - s_1|^{3/2} \right) - \frac{1}{2\delta} (|x_1 - z_1|^2 + (t_1 - s_1)^2). \end{aligned}$$

By applying Young inequality, it is clear that

$$|x_1 - x_0 + z - z_1|^3 + |t_1 - t_0 + s - s_1|^{3/2} \leq 2|x_1 - x_0|^3 + 2|z - z_1|^3 + 2|t_1 - t_0|^{3/2} + 2|s - s_1|^{3/2}.$$

After rearranging the terms, this yields

$$\begin{aligned} \sup_{(x, t) \in \Omega, t \leq t_1} \left( \phi(x, t) - \frac{1}{2\delta} (|x - z|^2 + (t - s)^2) \right) \\ \geq u(x_0, t_0) + b(t_1 - t_0) + \xi \cdot (x_1 - x_0) + \frac{1}{2}(x_1 - x_0) \cdot A(x_1 - x_0) - \frac{1}{3}\kappa \left( |x_1 - x_0|^3 + |t_1 - t_0|^{3/2} \right) \\ + b(s - s_1) + (\xi - p + A(x_1 - x_0)) \cdot (z - z_1) + \frac{1}{2}(z - z_1) \cdot A(z - z_1) \\ - \frac{1}{3}\kappa \left( |z - z_1|^3 + |s - s_1|^{3/2} \right) - \frac{1}{2\delta} (|x_1 - z_1|^2 + (t_1 - s_1)^2). \end{aligned}$$

Inserting this into (4.2.16) and after rearranging the terms, we obtain

$$\begin{aligned}
 & u(x_1, t_1) - u(x_0, t_0) - b(t_1 - t_0) - \xi \cdot (x_1 - x_0) - \frac{1}{2}(x_1 - x_0) \cdot A(x_1 - x_0) \\
 & \quad + \frac{1}{3}\kappa \left( |x_1 - x_0|^3 + |t_1 - t_0|^{3/2} \right) - v(z_1, s_1) \\
 & \geq \sup_{(z,s) \in \Gamma_\varepsilon, s \leq s_1} \left[ b(s - s_1) + (\xi - p - q + A(x_1 - x_0)) \cdot (z - z_1) \right. \\
 & \quad \left. + \frac{1}{2}(z - z_1) \cdot A(z - z_1) - \frac{1}{3}\kappa \left( |z - z_1|^3 + |s - s_1|^{3/2} \right) - v(z, s) \right] \\
 & \geq \sup_{\substack{(z,s) \in \Gamma_\varepsilon, s \leq s_1 \\ |z - z_1| \leq 2R_0\varepsilon}} \left[ b(s - s_1) + (\xi - p - q + A(x_1 - x_0)) \cdot (z - z_1) \right. \\
 & \quad \left. + \frac{1}{2}(z - z_1) \cdot A(z - z_1) - \frac{1}{3}\kappa \left( |z - z_1|^3 + |s - s_1|^{3/2} \right) - v(z, s) \right]. \quad (4.2.17)
 \end{aligned}$$

For the last equality, we restrained the sup to the points  $z$  near  $z_1$  which satisfy  $|z - z_1| \leq 2R_0\varepsilon$ . Moreover, the inequality  $|z - z_1|^3 \leq 2R_0\varepsilon|z - z_1|^2$  holds true under this condition and we define

$$\begin{aligned}
 \tilde{\psi}(z, s) &:= b(s - s_1) + (\xi - p - q + A(x_1 - x_0)) \cdot (z - z_1) \\
 & \quad + \frac{1}{2}(z - z_1) \cdot A(z - z_1) - \frac{2}{3}R_0\varepsilon\kappa|z - z_1|^2 - \frac{1}{3}\kappa|s - s_1|^{3/2}.
 \end{aligned}$$

Since  $(x_0, t_0) \in Q_\rho^f(x_1, t_1)$ , the point  $(x_1, t_1)$  satisfies  $t_1 \leq t_0$  and we can apply the  $W^{3,\sigma}$  estimate at the point  $(x_1, t_1)$  to get

$$\begin{aligned}
 0 & \leq u(x_1, t_1) - u(x_0, t_0) - b(t_1 - t_0) - \xi \cdot (x_1 - x_0) - \frac{1}{2}(x_1 - x_0) \cdot A(x_1 - x_0) + \frac{1}{2}\kappa \left( |x_1 - x_0|^3 + |t_1 - t_0|^{3/2} \right) \\
 & \leq \frac{1}{2}\kappa(|x_1 - x_0|^3 + |t_1 - t_0|^{3/2}) \leq \kappa\rho^3.
 \end{aligned}$$

By using the previous estimate and noticing that  $\psi(z_1, s_1) = 0$ , (4.2.17) yields

$$\tilde{\psi}(z_1, s_1) - v(z_1, s_1) + \kappa\rho^3 \geq \sup_{\substack{(z,s) \in \Gamma_\varepsilon, s \leq s_1 \\ |z - z_1| \leq 2R_0\varepsilon}} \left\{ \tilde{\psi}(z, s) - v(z, s) \right\}. \quad (4.2.18)$$

If we define the function  $\tilde{\psi} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  given by

$$\psi(z, s) := \tilde{\psi}(z, s) + \kappa\rho^3\chi\left(\frac{z - z_1}{\varepsilon}, \frac{s - s_1}{\varepsilon^2}\right).$$

where  $\chi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  is a  $C_b^3$  bump function satisfying the following properties:

$$\begin{cases} \text{Supp } \chi \subseteq [-1, 1]^{d+1}, \\ 0 \leq \chi \leq 1, \\ \chi(0, 0) = 1. \end{cases} \quad (4.2.19)$$

In particular, this implies that

$$\chi(x, t) = \mathbf{1}_{0,0}(x, t) \quad \text{for all } (x, t) \in \varepsilon\mathbb{Z}^d \times \varepsilon^2\mathbb{Z}.$$

Then the relationship (4.2.18) leads to

$$\psi(z_1, s_1) - v(z_1, s_1) = \sup_{\substack{(z,s) \in \Gamma_\varepsilon, s \leq s_1 \\ |z - z_1| \leq 2R_0\varepsilon}} \left\{ \psi(z, s) - v(z, s) \right\}. \quad (4.2.20)$$

In particular, the application  $z \rightarrow (\psi - v)(z, s_1)$  attains a global maximum at  $z_1$ . By applying Lemma 4.1.2, we get

$$F_\varepsilon[\psi(\cdot, s_1)](z_1) \geq F_\varepsilon[v(\cdot, s_1)](z_1).$$

In addition, we also deduce from (4.2.20) that

$$\tilde{\psi}(z_1, s_1) - v(z_1, s_1) \geq \tilde{\psi}(z_1, s_1 - \varepsilon^2) - v(z_1, s_1 - \varepsilon^2)$$

which implies

$$\hat{\partial}_t^\varepsilon \psi(z_1, s_1) \geq \hat{\partial}_t^\varepsilon v(z_1, s_1).$$

Since  $v$  is the solution of the scheme, this implies

$$0 = \hat{\partial}_t^\varepsilon v(z_1, s_1) + F_\varepsilon[v(\cdot, s_1)](z_1).$$

By combining the three last relationships, we get

$$\hat{\partial}_t^\varepsilon \psi(z_1, s_1) + F_\varepsilon[\psi(\cdot, s_1)](z_1) \geq 0. \quad (4.2.21)$$

*Step 6.* A direct computation gives

$$\hat{\partial}_t^\varepsilon \psi(z_1, s_1) = \frac{1}{\varepsilon^2} \left( \kappa \rho^3 + b \varepsilon^2 + \frac{1}{6} \kappa \varepsilon^3 \right) = b + \frac{1}{6} \kappa \varepsilon + \kappa \frac{\rho^3}{\varepsilon^2}.$$

The consistency estimate gives

$$F_\varepsilon[\psi(\cdot, s_1)](z_1) = \mathcal{F}(D^2 \psi(z_1, s_1)) = \mathcal{F}(A - \frac{1}{3} R_0 \kappa \varepsilon I_d).$$

By inserting these relations in (4.2.21), we obtain

$$b + \frac{1}{6} \kappa \varepsilon + \kappa \frac{\rho^3}{\varepsilon^2} + \mathcal{F}(A - \frac{4}{3} R_0 \kappa \varepsilon I_d) \geq 0.$$

By Step 1, we have  $b + \mathcal{F}(A) = -\frac{1}{2} E^\beta$ . By combining the choice  $\rho = \varepsilon + \delta^{1/2}$  with the uniform ellipticity of the operator  $\mathcal{F}$  with ellipticity constants  $0 < \lambda \leq 1$ , we have

$$C \frac{\delta^{3/2}}{\varepsilon^2} \kappa \geq \left[ \frac{4}{3} R_0 + 2 + \left( 1 + \frac{\delta^{1/2}}{\varepsilon} \right)^3 \right] \kappa \varepsilon \geq \frac{1}{2} E^\beta.$$

By inserting the expressions of  $\delta$  and  $\kappa$  given by (4.2.8) and (4.2.12), we deduce that

$$C E^{\frac{3}{2}(\frac{4}{\sigma} - 1) - \frac{8d+8}{\sigma^2} - \beta} \geq \varepsilon^2$$

Recalling that  $\varepsilon \leq C E^{1/\eta}$  and selecting

$$\eta < \left( \frac{4d+4}{\sigma^2} - \frac{6}{\sigma} + \frac{3}{4} \right)^{-1}$$

the particular choice for  $\beta$  given by

$$\beta > \frac{2}{\eta} + \frac{3}{2} \left( \frac{4}{\sigma} - 1 \right) - \frac{8d+8}{\sigma^2} > 0$$

yields the desired contradiction.  $\square$



### 4.3 Proof of the maximum principle in the doubled variables

In this section we give the argument for the double-variable maximum principle, Proposition 4.2.3. The proof is based on ABP-type estimates in both the continuous and discrete settings. We state the latter in the following two lemmas.

**Lemma 4.3.1.** *Suppose that  $\Omega \subseteq \mathbb{R}^{d+1}$ ,  $R > 0$ ,  $Q_{4R} \subseteq \Omega$  and  $u \in C(\Omega)$  satisfies*

$$\partial_t u + \mathcal{P}_{1,\Lambda}^+(D^2 u) \geq -1 \quad \text{in } \Omega.$$

*Assume that  $(x_0, t_0), (x_1, t_1) \in Q_R \cap \{u = \Gamma_u\}$  and  $(p_i, h_i) \in \partial\Gamma_u(x_i, t_i)$  for  $i = 0, 1$ . Then there exists  $C(d, \Lambda) > 1$  such that*

$$|p_1 - p_0|^2 + |h_1 - h_0|^2 \leq C(|x_1 - x_0|^2 + |t_1 - t_0|^2 + |x_1 - x_0|^4).$$

**Lemma 4.3.2.** *Suppose that  $\mathcal{F}$  satisfies (F1)-(F2) and  $F_\varepsilon$  satisfies (S2),  $\Lambda \subseteq \mathbb{R}^{d+1}$ ,  $R > 1$ ,  $Q_{4R} \cap \mathbb{Z}^{d+1} \subseteq \Lambda$  and  $u : \Lambda \rightarrow \mathbb{R}$  satisfies*

$$\hat{\partial}_t^\varepsilon u + F_\varepsilon[u] \geq -1 \quad \text{in } \Lambda.$$

*Assume, for each  $i \in \{0, 1\}$ , that  $(z_i, t_i) \in Q_R \cap \Lambda \cap \{u = \Gamma_u\}$  and  $(p_i, h_i) \in \partial\Gamma_u(z_i, t_i)$ . Then there exists  $C(d, \Lambda, K_0, K_1) > 1$  such that  $R \geq C\varepsilon$  implies that*

$$|p_1 - p_0|^2 + |h_1 - h_0|^2 \leq C(\varepsilon^2 + |z_1 - z_0|^2 + |t_1 - t_0|^2 + |z_1 - z_0|^4).$$

The proofs of these lemmas are found below. We next present the proof of Proposition 4.2.3.

**Proof of Proposition 4.2.3.** Suppose that, for each  $i \in \{1, 2\}$ ,

$$(x_i, t_i, y_i, s_i, p_i, q_i, h_i, k_i) \in \Omega \times \Gamma_\varepsilon \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$$

satisfies

$$0 = \Phi_{p_i, q_i, h_i, \hat{k}_i}(x_i, t_i, y_i, s_i) = \sup \left\{ \Phi_{p_i, q_i, h_i, \hat{k}_i}(\xi, \tau, \eta, \sigma) : (\xi, \tau, \eta, \sigma) \in \Omega \times \Gamma_\varepsilon, \tau \leq t_i \right\} \quad (4.3.1)$$

with

$$\left| \hat{k}_i - k_i \right| \leq \|u\|_{C^{0,\alpha}(\overline{\Omega})} \varepsilon^\alpha + \delta^{-1/2} \varepsilon^2.$$

Denote

$$r := (|x_1 - x_2|^2 + |y_1 - y_2|^2 + |t_1 - t_2| + |s_1 - s_2|)^{1/2}.$$

The main step in the argument is to show that, for some  $C(d, \Lambda, K_0) > 1$ ,

$$|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2 \leq C\delta^{-2}(\varepsilon^2 + r^2). \quad (4.3.2)$$

For  $i \in \{1, 2\}$ , we introduce some intermediate quantities  $H_i$  and  $K_i$  at the touching points given by

$$\begin{aligned} H_i &:= u(x_i, t_i) - p_i \cdot x_i - \frac{1}{2\delta} (|x_i - y_i|^2 + (t_i - s_i - h_i)^2) \\ &= \sup_{(\xi, \tau) \in \Omega, \tau \leq t} \left( u(\xi, \tau) - p_i \cdot \xi - \frac{1}{2\delta} (|\xi - y_i|^2 + (\tau - s_i - h_i)^2) \right) \end{aligned} \quad (4.3.3)$$

and

$$\begin{aligned} K_i &:= -v(y_i, s_i) - \frac{1}{2\delta} (|x_i - y_i|^2 + (t_i - s_i - h_i)^2) - q_i \cdot y_i \\ &= \sup_{(\eta, \sigma) \in \Gamma_\varepsilon} \left( -v(\eta, \sigma) - q_i \cdot \eta - \frac{1}{2\delta} (|x_i - \eta|^2 + (t_i - \sigma - h_i)^2) \right). \end{aligned} \quad (4.3.4)$$

Notice that  $\hat{k}_i$  can be expressed by the relation

$$\hat{k}_i = H_i + K_i + \frac{1}{2\delta}(|x_i - y_i|^2 + (t_i - s_i)^2) \quad (4.3.5)$$

The first three steps contain the proof of (4.3.2) and in the final step we deduce the proposition from (4.3.2). Throughout we let  $C$  and  $c$  denote positive constants which may vary from line to line and depend only on  $(d, \Lambda, K_0)$ .

We may assume without loss of generality that

$$t_1 - h_1 \leq t_2 - h_2. \quad (4.3.6)$$

*Step 1.* We show there exists a constant  $C$  depending on  $\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}, |p|, |q|$  and  $\text{diam } \Omega$  such that for each  $i \in \{1, 2\}$ ,

$$\max\{|x_i - y_i|, |t_i - s_i - h_i|\} \leq C\delta^{1/2}. \quad (4.3.7)$$

To prove this, we distinguish two cases. Assume first that  $s + h \leq t$ . we get by using the definition of  $H$  that

$$u(x, t) - p \cdot x - \frac{1}{2\delta}(|x - y|^2 + (t - s - h)^2) \geq u(y, s + h) - p \cdot y.$$

After rearranging the terms, we get

$$\begin{aligned} \frac{1}{2\delta}(|x - y|^2 + (t - s - h)^2) &\leq u(x, t) - u(y, s + h) - p \cdot (x - y) \\ &\leq 2\|u\|_{L^\infty(\Omega)} + 2|p|\text{diam } \Omega \end{aligned}$$

which yields the result. Now assume that  $t \leq s + h$ , we get by definition of  $K$  that

$$-v(y, s) - \frac{1}{2\delta}(|x - y|^2 + (t - s - h)^2) - q \cdot y \geq -v(x, t + h) - q \cdot x.$$

This implies

$$\begin{aligned} \frac{1}{2\delta}(|x - y|^2 + (t - s - h)^2) &\leq v(x, t + h) - v(y, s) + q \cdot (x - y) \\ &\leq 2\|v\|_{L^\infty(\Omega)} + 2|q|\text{diam } \Omega, \end{aligned}$$

By putting together the two cases, we get (4.3.7).

*Step 2.* We claim that

$$|p_1 - p_2|^2 + |H_1 - H_2|^2 \leq C\delta^{-2}(1 + r^2)^3 r^2. \quad (4.3.8)$$

We are going to distinguish two cases depending on the sign of  $s_1 + h_1 - (s_2 + h_2)$ .

Consider the first alternative for which  $s_1 + h_1 \leq s_2 + h_2$ . In that case, define

$$\tilde{u}(x, t) := -u(x, t) + \frac{1}{2\delta}(|x - y_1|^2 + (t - s_1 - h_1)^2).$$

Let  $T_m := \min\{s_1, t_1, t_2\}$  and  $T_M := \max\{s_2, t_1, t_2\}$ . Note that  $T_M - T_m \leq C + r$ . By (4.2.2) and Step 1, we find that, for  $\rho := C(1 + r)$ , we have  $(x_i, t_i), (y_i, s_i) \in Q_\rho(x_1, T_M)$  for each  $i \in \{1, 2\}$  and

$$\begin{aligned} \partial_t \tilde{u} + \mathcal{P}_{1, \Lambda}^+(D^2 \tilde{u}) &= -\partial_t u + \delta^{-1}(t - s_1 - h_1) + \mathcal{P}_{\lambda, \Lambda}^+ \left( -D^2 u + \frac{1}{\delta} I_d \right) \\ &\geq -C\delta^{-1} \quad \text{in } Q_\rho(x_1, T_M). \end{aligned} \quad (4.3.9)$$

According to (4.3.1) with  $i = 1$ , we have

$$\tilde{u}(x_1, t_1) = \Gamma_{\tilde{u}}(x_1, t_1) \quad \text{and} \quad (-p_1, \tilde{u}(x_1, t_1) + p_1 \cdot x_1) \in \partial \Gamma_{\tilde{u}}(x_1, t_1). \quad (4.3.10)$$

Note that  $H_1 = -(\tilde{u}(x_1, t_1) + p_1 \cdot x_1)$ . We next check that (4.3.1) with  $i = 2$  implies

$$\left(-p_2 + \frac{y_2 - y_1}{\delta}, \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2\right) \in \partial\Gamma_{\tilde{u}}(x_2, t_2). \quad (4.3.11)$$

To see this, note that the former implies that the map

$$(x, t) \mapsto u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x$$

achieves its supremum over  $\Omega \cap \{t \leq t_2\}$  at  $(x_2, t_2)$ , and we may write this as

$$\begin{aligned} u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x \\ = -\tilde{u}(x, t) - \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x - \frac{1}{\delta} (s_1 + h_1 - (s_2 + h_2)) t \\ + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 + (s_2 + h_2)^2). \end{aligned}$$

Using that  $s_1 + h_1 \leq s_2 + h_2$ , we obtain

$$\begin{aligned} \inf_{(x,t) \in \Omega, t \leq t_2} \left[ \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t \right] \\ \leq \inf_{(x,t) \in \Omega, t \leq t_2} \left[ \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t \right] \\ = \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t_2, \end{aligned}$$

that is,

$$(x, t) \mapsto \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x \text{ attains its infimum on } \Omega \cap \{t \leq t_2\} \text{ at } (x_2, t_2).$$

This yields (4.3.11). Applying Lemma 4.3.1 in view of (4.3.15), (4.3.10) and (4.3.11) yields

$$\begin{aligned} \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + \left| \tilde{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \tilde{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \right|^2 \\ \leq C\delta^{-2} (|x_1 - x_2|^2 + |t_1 - t_2|^2), \quad (4.3.12) \end{aligned}$$

First, we have

$$|p_1 - p_2|^2 \leq 2 \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + 2\delta^{-2} |y_2 - y_1|^2 \leq C\delta^{-2} (|x_1 - x_2|^2 + |y_2 - y_1|^2 + |t_1 - t_2|^2).$$

It remains to estimate  $|H_1 - H_2|^2$ . We compute

$$\begin{aligned} \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 \\ = -u(x_2, t_2) + \frac{1}{2\delta} (|x_2 - y_1|^2 + (t_2 - s_1 - h_1)^2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 \\ = -H_2 + \frac{1}{2\delta} (|x_2 - y_1|^2 + (t_2 - s_1 - h_1)^2 - |x_2 - y_2|^2 - (t_2 - s_2 - h_2)^2) - \frac{y_2 - y_1}{\delta} \cdot x_2 \\ = -H_2 + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 - (s_2 + h_2)^2 - 2(s_1 + h_1 - s_2 - h_2)t_2) \quad (4.3.13) \end{aligned}$$

We have

$$\begin{aligned}
 & \left| |y_1|^2 - |y_2|^2 + (s_1 + h_1)^2 - (s_2 + h_2)^2 - 2(s_1 - s_2 + h_1 - h_2)t_2 \right|^2 \\
 & \leq 3 \left| |y_1|^2 - |y_2|^2 \right|^2 + 3 \left| (s_1 + h_1)^2 - (s_2 + h_2)^2 \right|^2 + 12 \left| (s_1 - s_2 + h_1 - h_2)t_2 \right|^2 \\
 & \leq C(\text{diam } \Omega)^2 |y_1 - y_2|^2 + \left( |s_1 + h_1 + s_2 + h_2|^2 + 12t_2^2 \right) |(s_1 + h_1) - (s_2 + h_2)|^2 \\
 & \leq C(\text{diam } \Omega)^2 |y_1 - y_2|^2 + C(\max\{h_1^2, h_2^2\} + T^2) ((t_1 - t_2)^2 + \delta) \\
 & \leq C(\text{diam } \Omega + \max\{|h_1|, |h_2|\} + T)^2 (\delta + |t_1 - t_2|^2 + |y_1 - y_2|^2). \tag{4.3.14}
 \end{aligned}$$

where in the third line, it was Step 1 that gave us

$$|(s_1 + h_1) - (s_2 + h_2)| \leq |t_1 - t_2| + C\delta^{1/2}.$$

Thanks to (4.3.10), (4.3.13), (4.3.14) and (4.3.12), we deduce that

$$\begin{aligned}
 |H_1 - H_2|^2 & \leq 2 \left| \tilde{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \tilde{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \right|^2 + 2L^2 \\
 & \leq C(1 + \text{diam } \Omega + \max\{|h_1|, |h_2|\} + T)^2 \delta^{-2} (\delta + |x_1 - x_2|^2 + |t_1 - t_2|^2 + |y_1 - y_2|^2).
 \end{aligned}$$

We now consider the second alternative for which  $s_1 + h_1 > s_2 + h_2$ . Define

$$\hat{u}(x, t) := -u(x, t) + \frac{1}{2\delta} (|x - y_1|^2 + (t - s_1 - h_1)^2) + \left( \frac{s_1 + h_1 - s_2 - h_2}{\delta} \right) t.$$

We compute

$$\begin{aligned}
 \partial_t \hat{u} + \mathcal{P}_{1,\Lambda}^+(D^2 \hat{u}) & = -\partial_t u + \delta^{-1}(t - s_2 - h_2) + \mathcal{P}_{\lambda,\Lambda}^+ \left( -D^2 u + \frac{1}{\delta} I_d \right) \\
 & \geq -C\delta^{-1} \quad \text{in } Q_\rho(x_1, T_M).
 \end{aligned} \tag{4.3.15}$$

Using (4.3.10) and  $s_1 + h_1 > s_2 + h_2$ , we deduce that

$$\hat{u}(x_1, t_1) = \Gamma_{\hat{u}}(x_1, t_1) \quad \text{and} \quad (-p_1, \hat{u}(x_1, t_1) + p_1 \cdot x_1) \in \partial \Gamma_{\hat{u}}(x_1, t_1). \tag{4.3.16}$$

We get that

$$\begin{aligned}
 u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x \\
 = -\hat{u}(x, t) - \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 + (s_2 + h_2)^2).
 \end{aligned}$$

that is,

$$(x, t) \mapsto \hat{u}(x, t) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x \text{ attains its infimum on } \Omega \cap \{t \leq t_2\} \text{ at } (x_2, t_2).$$

This yields (4.3.11). Applying Lemma 4.3.1 in view of (4.3.10), (4.3.11) and (4.3.15) yields

$$\begin{aligned}
 \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + \left| \hat{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \hat{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \right|^2 \\
 \leq C\delta^{-2} (|x_1 - x_2|^2 + |t_1 - t_2|^2),
 \end{aligned}$$

*Step 3.* We show that

$$|q_1 - q_2|^2 + |K_1 - K_2|^2 \leq C\delta^{-2}(\varepsilon^2 + \delta + (1 + r^2)r^2). \tag{4.3.17}$$

Define

$$\tilde{v}(y, s) := v(y, s) + \frac{1}{2\delta} (|y - x_1|^2 + (t_1 - s - h_1)^2).$$

According to (S3), we have

$$F_\varepsilon[\tilde{v}] \geq F_\varepsilon[v] - C\delta^{-1}.$$

Therefore, by using Step 1,  $\tilde{v}$  satisfies

$$\widehat{\partial}_t^\varepsilon \tilde{v} + F_\varepsilon[\tilde{v}] \geq \widehat{\partial}_t^\varepsilon v + \frac{1}{\delta}(s + h_1 - t_1) - \frac{1}{2\delta} + F_\varepsilon[v] - C\delta^{-1} \geq -C\delta^{-1} \quad \text{in } Q_{4r}(y_1, T_M), \quad (4.3.18)$$

where  $T_M := \max\{s_1, s_2, t_2\}$ . According to (4.3.1) with  $i = 1$ , we have

$$\tilde{v}(y_1, s_1) = \Gamma_{\tilde{v}}(y_1, s_1) \quad \text{and} \quad (-q_1, \tilde{v}(y_1, s_1) + q_1 \cdot y_1) \in \partial\Gamma_{\tilde{v}}(y_1, s_1). \quad (4.3.19)$$

We next check that (4.3.1) for  $i = 2$  implies that

$$\left(-q_2 - \frac{x_1 - x_2}{\delta}, \tilde{v}(y_2, s_2) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y_2\right) \in \partial\Gamma_{\tilde{v}}(y_2, s_2). \quad (4.3.20)$$

We observe that

$$(y, s) \mapsto v(y, s) + \frac{1}{2\delta} (|y - x_2|^2 + (t_2 - h_2 - s)^2) + q_2 \cdot y$$

achieves its infimum over  $\Lambda \cap \{s \leq s_2\}$  at  $(y_2, s_2)$  and thus, using the identity

$$\begin{aligned} v(y, s) + \frac{1}{2\delta} (|y - x_2|^2 + (t_2 - h_2 - s)^2) + q_2 \cdot y \\ = \tilde{v}(y, s) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y + \frac{t_1 - h_1 - t_2 + h_2}{\delta} s \\ + \frac{1}{2\delta} (|x_2|^2 + (t_2 - h_2)^2 - |x_1|^2 - (t_1 - h_1)^2) \end{aligned}$$

and that  $t_1 - h_1 \leq t_2 - h_2$  by (4.4.7), we find

$$\begin{aligned} \inf_{(y,s) \in \Lambda, s \leq s_2} \left[ \tilde{v}(y, s) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y + \frac{t_1 - h_1 - t_2 + h_2}{\delta} s_2 \right] \\ \leq \inf_{(y,s) \in \Lambda, s \leq s_2} \left[ \tilde{v}(y, s) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y + \frac{t_1 - h_1 - t_2 + h_2}{\delta} s \right] \\ = \tilde{v}(y_2, s_2) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y_2 + \frac{t_1 - h_1 - t_2 + h_2}{\delta} s_2, \end{aligned}$$

that is,

$$(y, s) \mapsto \tilde{v}(y, s) + \left(q_2 + \frac{x_1 - x_2}{\delta}\right) \cdot y \text{ attains its infimum over } \Lambda \cap \{s \leq s_2\} \text{ at } (y_2, s_2).$$

Applying Lemma 4.3.2 in view of (4.3.18), (4.3.19) and (4.3.20) yields

$$\begin{aligned} \left| q_1 - q_2 + \frac{x_2 - x_1}{\delta} \right|^2 + \left| \tilde{v}(y_1, s_1) + q_1 \cdot y_1 - \left( \tilde{v}(y_2, s_2) + \left( q_2 + \frac{x_2 - x_1}{\delta} \right) \cdot y_2 \right) \right|^2 \\ \leq C\delta^{-2} (\varepsilon^2 + |y_1 - y_2|^2 + |s_1 - s_2|^2 + |y_1 - y_2|^4) \leq C\delta^{-2} (\varepsilon^2 + r^2 + r^4). \end{aligned}$$

By doing similar computations than in Step 2, we get

$$|q_1 - q_2|^2 + |K_1 - K_2|^2 \leq C\delta^{-2} (\varepsilon^2 + \delta + (1 + r^2)r^2).$$

*Step 4.* We claim that

$$|k_1 - k_2|^2 \leq C\delta^{-2}(\varepsilon^2 + \varepsilon^{2\alpha} + \delta + (1 + r^2)r^2).$$

First by using (4.3.5) we deduce

$$|\hat{k}_1 - \hat{k}_2| \leq |H_1 - H_2| + |K_1 - K_2| + \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1)^2 - |x_2 - y_2|^2 - (t_2 - s_2)^2)$$

Then the claim follows directly from Steps 2-3 and the inequality  $|k_i - \hat{k}_i| \leq (C\varepsilon^\alpha + \varepsilon^2\delta^{-1/2})$ . By combining Steps 2-3, we obtain

$$|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2 \leq C\delta^{-2}(\varepsilon^2 + \varepsilon^{2\alpha} + \delta + (1 + r^2)r^2).$$

We get that

$$(|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2)^{1/2} \leq C\delta^{-1}(\varepsilon^\alpha + \delta^{1/2} + (1 + r^2)^{1/2}r).$$

Since  $r \leq \text{diam } \Omega$ , there is a constant  $C$  depending on  $T$  and  $\text{diam } \Omega$  such that

$$(|p_1 - p_2|^2 + |h_1 - h_2|^2 + |a_1 - a_2|^2 + |b_1 - b_2|^2)^{1/2} \leq C\delta^{-1}(\varepsilon^\alpha + \delta^{1/2} + r).$$

By applying Lemma 4.3.3, we deduce that

$$|W| \leq C\delta^{-(2d+2)} \left(1 + \frac{\varepsilon^\alpha + \delta^{1/2}}{\rho}\right)^{2d+2} |\tilde{V}_\rho|.$$

which yields the desired inequality.  $\square$

The next lemma contains the measure argument we need to deduce the measure estimate between the set of the touching points and the set of parameters  $(p, q, h, k)$ .

**Lemma 4.3.3.** *Let  $H > 0$ ,  $E \subseteq \mathbb{R}^m$ ,  $m \geq 1$ , be bounded and  $\phi : \mathbb{R}^m \mapsto \mathbb{R}^m$  such that for all  $x, y \in E$ ,*

$$|\phi(x) - \phi(y)| \leq L|x - y| + H.$$

*Then there exists a constant  $C > 1$  depending only on the dimension  $m$ , such that, for all  $\rho > 0$ , we have*

$$|\phi(E)| \leq C \left(L + \frac{H}{\rho}\right)^m |E + B_\rho|.$$

*Proof.* Set  $\tilde{E}_\rho := B_\rho + E$ . For  $\rho > 0$ , we consider the collection  $\{B_\rho(x)\}_{x \in E}$  which satisfies in particular  $B_\rho(x) \subseteq \tilde{E}_\rho$  for all  $x \in E$ . By applying Vitali's covering lemma to the closure of  $\tilde{E}_\rho$  which is compact, we may extract a finite subcollection  $\{B_\rho(x_i) : 1 \leq i \leq N\}$  such that the  $B_\rho(x_i)$  are disjoint and

$$\bigcup_{i=1}^N B_\rho(x_i) \subseteq \tilde{E}_\rho \subseteq \bigcup_{i=1}^N B_{5\rho}(x_i).$$

In particular, this implies that  $|\tilde{E}_\rho| \geq N\rho^m$ . For all  $y \in B_{5\rho}(x_i)$ ,

$$|\phi(y) - \phi(x_i)| \leq L|y - x_i| + H \leq 5\rho L + H.$$

and this implies  $\phi(B_{5\rho}(x_i)) \subseteq B_{5L\rho+H}(z_i)$ , where we set  $z_i := \phi(x_i)$ . Since  $E \subseteq \tilde{E}_\rho$ , we deduce

$$\phi(E) \subseteq \phi(\tilde{E}_\rho) \subseteq \bigcup_{i=1}^N \phi(B_{5\rho}(x_i)) \subseteq \bigcup_{i=1}^N B_{5L\rho+H}(z_i).$$

Taking the Lebesgue measure of these sets, we obtain

$$|\phi(E)| \leq \sum_{i=1}^N |B_{5L\rho+H}(z_i)| = N(5L\rho + H)^m \leq \left(5L + \frac{H}{\rho}\right)^m |\tilde{E}_\rho|,$$

which yields the desired estimate.  $\square$

**Proof of Lemma 4.3.1.** Throughout,  $C$  and  $c$  denote positive constants which may change in each occurrence and depend only on  $d$  and  $\Lambda$ . By a covering argument and rescaling, it suffices to consider the case  $R = 2$  and  $0 < r \leq c$ . We assume without loss of generality that  $t_0 \geq t_1$ . The lemma is immediate from Young's inequality and the following two estimates:

$$|h_0 - h_1| \leq C \left( |t_0 - t_1| + |p_0 - p_1| \max\{|x_1 - x_0|, |t_0 - t_1|^{1/2}\} \right) \quad (4.3.21)$$

and

$$|p_0 - p_1| \leq C|x_1 - x_0|. \quad (4.3.22)$$

Denote  $r^2 = |x_1 - x_0|^2 + |t_1 - t_0|^2$  and  $r_{\text{par}}^2 = |x_1 - x_0|^2 + |t_1 - t_0|$ . Under (4.3.21) and (4.3.22), we obtain the parabolic estimate

$$|p_0 - p_1|^2 + |h_0 - h_1| \leq C(|x_1 - x_0|^2 + |t_1 - t_0|) = Cr_{\text{par}}^2,$$

and

$$|p_0 - p_1|^2 + |h_0 - h_1|^2 \leq C(|x_1 - x_0|^2 + |t_1 - t_0|^2 + |x_1 - x_0|^4) \leq C(1 + r^2)r^2.$$

*Step 1.* We prove (4.3.21). By translating, we may assume  $(x_0, t_0) = (0, 0)$  and, by subtracting a plane from  $u$ , that  $p_0 = 0$  and  $u(0, 0) = 0$ . This implies  $h_0 = 0$ ,  $\Gamma_u(0, 0) = 0$  and

$$\begin{cases} u \geq \Gamma_u \geq 0 & \text{on } Q_4 \cap \{(x, s) : t_1 \leq s \leq 0\}, \\ u(x, t) \geq \Gamma_u(x, t) \geq \max\{0, h_1 + p_1 \cdot x\} & \text{for } (x, t) \in Q_4, t \leq t_1. \end{cases}$$

In particular,  $u(x_1, t_1) \geq 0$  and thus

$$h_1 - h_0 = h_1 = u(x_1, t_1) - p_1 \cdot x_1 \geq -|p_1||x_1|. \quad (4.3.23)$$

Thus to obtain (4.3.21), we have left to obtain an upper bound on  $h_1$ . Observe that, if  $t_1 = 0$ , then  $u(0, 0) = 0$  implies  $h_1 \leq 0$ , and so we are done. Thus we may assume that  $t_1 < 0$ . Arguing by contradiction, we suppose that

$$h_1 > (2 + \Lambda)(|t_1| + |p_1||t_1|^{1/2}). \quad (4.3.24)$$

Consider the function  $\varphi(y, s) := |t_1| - (1 + \Lambda)s - |y|^2$  and observe that

$$\partial_t \varphi + \mathcal{P}_{1, \Lambda}^+(D^2 \varphi) = -1 \quad \text{in } \mathbb{R}^{d+1}.$$

Applying the comparison principle in the cylinder  $\tilde{Q} := B_{((2+\Lambda)|t_1|)^{1/2}}(0, 0) \times [t_1, 0] \subseteq Q_R$ , we find

$$\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial_p \tilde{Q}} (u - \varphi) \leq (u - \varphi)(0, 0) < 0. \quad (4.3.25)$$

Since  $\varphi$  is decreasing in time, for every  $(x, t) \in \partial_p \tilde{Q} \cap \{(y, s) : t_1 < s \leq 0\}$ ,

$$\varphi(x, t) \leq \varphi(x, t_1) = |t_1| + (1 + \Lambda)|t_1| - |x|^2 \leq 0 \leq \Gamma_u(x, t).$$

Here we used the fact that  $-t_1 = |t_1| \leq r^2$ . By the hypothesis (4.3.24), we have that, for every  $(x, t) \in \partial_p \tilde{Q} \cap \{(y, s) : s = t_1\}$ ,

$$\begin{aligned} \varphi(x, t) &\leq \varphi(0, t_1) = |t_1| + (1 + \Lambda)|t_1| \leq (2 + \Lambda)|t_1| \\ &\leq h_1 - (2 + \Lambda)^{1/2}|p||t_1|^{1/2} \leq h_1 - p_1 \cdot x \leq \Gamma_u(x, t). \end{aligned}$$

The previous two lines yield  $\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \geq 0$  which contradicts (4.3.25). We deduce that  $h_1 \leq (2 + \Lambda)(r^2 + |p_1|r) \leq C(r^2 + |p_1|r)$ . This completes the proof of (4.3.21).

*Step 2.* We prove (4.3.22). Contrary to Step 1, here we assume by translating that  $(x_1, t_1) = (0, 0)$  and, by subtracting a plane from  $u$ , that  $p_1 = 0$  and  $u(x_1, t_1) = 0$ . We have

$$\begin{cases} u(x, t) \geq \Gamma_u(x, t) \geq h_0 + p_0 \cdot x & \text{for } (x, t) \in Q_4, \ 0 \leq t \leq t_0, \\ u(x, t) \geq \Gamma_u(x, t) \geq \max\{0, h_0 + p_0 \cdot x\} & \text{for } (x, t) \in Q_4, \ t \leq 0. \end{cases}$$

It is convenient to take  $e := p_0/|p_0|$ . With  $A, \sigma > 1$  to be selected below, we consider the test function

$$\varphi(x, t) := t + \frac{1}{2}Ar(x \cdot e) + \frac{1}{2}\sigma(x \cdot e)^2 - \frac{3}{2}|x - (x \cdot e)e|^2.$$

After a computation, we find that the choice  $\sigma := 2 + 3\Lambda(d-1) \leq C$  leads to

$$\partial_t \varphi + \mathcal{P}_{1,\Lambda}^+(D^2 \varphi) = (1 + 3\Lambda(d-1)) - \sigma = -1 \quad \text{in } \mathbb{R}^{d+1}.$$

The comparison principle applied to  $u$  and  $\varphi$  with respect to the domain

$$\tilde{Q} := \{x \in \mathbb{R}^d : |x - (x \cdot e)e| \leq 1, \ |x \cdot e| \leq \beta r\} \times [-1, 0] \subseteq Q_R$$

with  $\beta > 1$  to be selected below, yields

$$\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial_p \tilde{Q}} (u - \varphi) = \inf_{\tilde{Q}} (u - \varphi) \leq (u - \varphi)(0, 0) = 0.$$

Select  $(y, t) \in \partial_p \tilde{Q}$  such that  $(\Gamma_u - \varphi)(y, t) \leq 0$ . There are four possibilities to consider: (i)  $y \cdot e = \beta r$ , (ii)  $y \cdot e = -\beta r$ , (iii)  $|y - (y \cdot e)e| = 1$ , and (iv)  $t = -1$ . Observe that, in each of the cases (i)–(iii), we have that  $t = 0$  by the monotonicity of  $\Gamma_u - \varphi$  in time. We argue that, if  $\beta$  and  $A$  are chosen appropriately, then cases (ii) – (iv) are impossible and case (i) leads to the bound (4.3.22).

In case (i), we find

$$h_0 + |p_0|\beta r \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) \leq \frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2.$$

Dividing by  $\beta r$ , using (4.3.21) and rearranging, we get

$$|p_0| \leq \frac{-h_0}{\beta r} + \frac{1}{2}Ar + \frac{1}{2}\sigma\beta r \leq \frac{C}{\beta}|p_0| + C(1 + A + \beta)r.$$

Choosing  $\beta := C$  yields  $|p_0| \leq C(1 + A)r \leq Cr$ , as desired, provided that  $A \leq C$ . Case (ii) is impossible if we select  $A := 2\sigma\beta \leq C$ , since it implies

$$0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) \leq -\frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2 < 0.$$

Case (iii) is also impossible if  $0 < r \leq c$ , since it implies

$$0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) \leq \frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2 - \frac{3}{2} \leq Cr^2 - \frac{3}{2} < 0.$$

Finally, case (iv) is impossible if  $0 < r \leq c$ , since

$$0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, -1) \leq -1 + \frac{1}{2}A\beta r^2 + \frac{1}{2}\sigma\beta^2 r^2 \leq -1 + Cr^2.$$

This completes the proof of (4.3.22) and the lemma.  $\square$

We next give the proof of Lemma 4.3.2, which is an adaptation of the proof of the previous lemma to the discrete case, using a discrete comparison principle and (S2) to justify the test function computations. At the risk of repetition, we give the complete details of the argument for the reader's convenience.



**Proof of Lemma 4.3.2.** Throughout,  $C$  and  $c$  denote positive constants which may change in each occurrence and depend only on  $d$  and  $\lambda$ . We assume without loss of generality that  $t_0 \geq t_1$  and denote  $r^2 := \varepsilon^2 + |z_0 - z_1|^2 + (t_0 - t_1)$ . By a covering argument and rescaling, it suffices to consider the case that  $0 < r \leq cR$ . The lemma is immediate from Young's inequality and the following two estimates:

$$|h_0 - h_1| \leq C(r^2 + |p_0 - p_1|r) \quad (4.3.26)$$

and

$$|p_0 - p_1| \leq Cr. \quad (4.3.27)$$

*Step 1.* We prove (4.3.26). By translating, we may assume  $(z_0, t_0) = (0, 0)$  and, by subtracting a plane from  $u$ , that  $p_0 = 0$  and  $u(0, 0) = 0$ . This implies  $h_0 = 0$ ,  $\Gamma_u(0, 0) = 0$  and

$$\begin{cases} u \geq \Gamma_u \geq 0 & \text{on } Q_4 \cap \{(z, s) \in \mathbb{Z}^{d+1} : t_1 \leq s \leq 0\}, \\ u(z, t) \geq \Gamma_u(z, t) \geq \max\{0, h_1 + p_1 \cdot z\} & \text{for } (z, t) \in Q_4 \cap \mathbb{Z}^{d+1}, t \leq t_1. \end{cases}$$

In particular,  $u(z_1, t_1) \geq 0$  and thus

$$h_1 - h_0 = h_1 = u(z_1, t_1) - p_1 \cdot x_1 \geq -|p_1||z_1 - z_0|. \quad (4.3.28)$$

This is the lower bound on  $h_1$ . To get an upper bound, we consider two cases:  $t_1 = 0$  and  $t_1 < 0$ . The first case implies  $u(0, 0) = 0$  and thus  $h_1 \leq 0$ , and we are done. Therefore we assume  $t_1 < 0$  and, arguing by contradiction, that

$$h_1 > (2 + \Lambda)(r^2 + |p_1|r) + \varepsilon|p_1|. \quad (4.3.29)$$

Using (S2) and  $\mathcal{F}(0) = 0$ , we find that the parabola  $\varphi(y, s) := (r^2 - (1 + \Lambda)s - |y|^2)$  satisfies

$$\hat{\partial}_t^\varepsilon \varphi(z, t) + F_\varepsilon[\varphi(\cdot, t)](z) = \partial_t \varphi(z, t) + \mathcal{F}(D^2 \varphi(z, t)) \leq \partial_t \varphi(z, t) + \mathcal{P}_{1, \Lambda}^+(D^2 \varphi(z, t)) = -1 \quad \text{in } \mathbb{R}^{d+1}.$$

We apply the discrete comparison principle in the cylinder

$$\tilde{Q} := (B_{r(2+\Lambda)^{1/2}}(0, 0) \times [t_1, 0]) \cap \mathbb{Z}^{d+1} \subseteq Q_R$$

to obtain

$$\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial_p \tilde{Q}} (u - \varphi) \leq (u - \varphi)(0, 0) < 0. \quad (4.3.30)$$

Since  $\varphi$  is decreasing in  $t$ , we have, for every  $(z, t) \in \partial_p \tilde{Q} \cap \{(y, s) : t_1 < s \leq 0\}$ ,

$$\varphi(z, t) \leq \varphi(z, t_1) \leq -(1 + \Lambda)(r^2 + t_1) \leq 0 \leq \Gamma_u(z, t).$$

Here we used the fact that  $-t_1 = |t_1| \leq r^2$ . Now we use (4.3.29) to find that, for every  $(z, t) \in \partial_p \tilde{Q} \cap \{(y, s) : s = t_1\}$ ,

$$\varphi(z, t) \leq \varphi(0, t_1) = r^2 + (1 + \Lambda)|t_1| \leq h_1 - (2 + \Lambda)|p_1|r - \varepsilon|p_1| \leq h_1 - p_1 \cdot z \leq \Gamma_u(z, t).$$

The previous two lines yield  $\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \geq 0$  which contradicts (4.3.30). We deduce that

$$h \leq (2 + \Lambda)(r^2 + |p_1|r) + \varepsilon|p_1| \leq C(r^2 + |p_1|r).$$

This completes the proof of (4.3.26).

*Step 2.* We prove (4.3.27). We may assume by translating that  $(z_1, t_1) = (0, 0)$  and, after subtracting a plane from  $u$ , that  $p_1 = 0$  and  $u(z_1, t_1) = 0$ . We have

$$\begin{cases} u(z, t) \geq \Gamma_u(z, t) \geq h_0 + p_0 \cdot z & \text{for } (z, t) \in Q_4, 0 \leq t \leq t_0, \\ u(z, t) \geq \Gamma_u(z, t) \geq \max\{0, h_0 + p_0 \cdot z\} & \text{for } (z, t) \in Q_4, t \leq 0. \end{cases}$$

It is convenient to take  $e := p_0/|p_0|$ . With  $A, \sigma > 1$  to be selected below, we consider the test function

$$\varphi(y, s) := s + \frac{1}{2}A(r + \varepsilon)(y \cdot e) + \frac{1}{2}\sigma(y \cdot e)^2 - \frac{3}{2}|y - (y \cdot e)e|^2.$$

We take  $\sigma := 2 + 3\Lambda(d - 1) \leq C$  and apply (S2) and  $\mathcal{F}(0) = 0$  to get

$$\hat{\partial}_t^\varepsilon \varphi + F_\varepsilon[\varphi] = \varphi_t + \mathcal{F}(D^2 \varphi) \leq \partial_t \varphi + \mathcal{P}_{1,\Lambda}^+(D^2 \varphi) = -1 \quad \text{in } \mathbb{Z}^{d+1}.$$

The comparison principle applied to  $u$  and  $\varphi$  with respect to the domain

$$\tilde{Q} := \mathbb{Z}^{d+1} \cap (\{x \in \mathbb{R}^d : |x - (x \cdot e)e| \leq R, |x \cdot e| \leq \beta r\} \times [-R^2, 0]) \subseteq Q_{2R}$$

with  $1 < \beta \leq r/R$  to be selected below, yields

$$\inf_{\partial_p \tilde{Q}} (\Gamma_u - \varphi) \leq \inf_{\partial_p \tilde{Q}} (u - \varphi) = \inf_{\tilde{Q}} (u - \varphi) \leq (u - \varphi)(0, 0) = 0.$$

Select  $(y, t) \in \partial_p \tilde{Q}$  such that  $(\Gamma_u - \varphi)(y, t) \leq 0$ . There are four possibilities to consider: (i)  $\beta r \leq y \cdot e \leq \beta r + \varepsilon$ , (ii)  $-\beta r - \varepsilon \leq y \cdot e \leq -\beta r$ , (iii)  $R \leq |y - (y \cdot e)e| \leq R + \varepsilon$ , and (iv)  $t \leq -R^2 + 1$ . Observe that, in each of the cases (i)–(iii), we have that  $t = 0$  by the monotonicity of  $\Gamma_u - \varphi$  in time. We show that, after choosing  $\beta$  and  $A$  appropriately, cases (ii)–(iv) are impossible and case (i) leads to the bound (4.3.27).

In case (i), we find

$$h_0 + |p_0|\beta r \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) \leq \frac{1}{2}A\beta(r + \varepsilon)^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon)^2.$$

Dividing by  $\beta r$ , using (4.3.26), that  $\varepsilon \leq r$  and rearranging, we get

$$|p_0| \leq \frac{|h_0|}{\beta r} + 2Ar + 2\sigma\beta r \leq \frac{C}{\beta}|p_0| + C(1 + A + \beta)r.$$

Choosing  $\beta := C$  yields  $|p_0| \leq C(1 + A)r \leq Cr$ , as desired, provided that we select  $A \leq C$ . Next, we observe that Case (ii) is impossible if we take  $A := 2\sigma\beta \leq C$ , since it implies

$$0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) \leq -\frac{1}{2}A\beta(r + \varepsilon)^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon)^2 < 0.$$

Case (iii) is also impossible if  $0 < r \leq cR$ , since it implies

$$\begin{aligned} 0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, 0) &\leq \frac{1}{2}A\beta(r + \varepsilon)^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon)^2 - \frac{3}{2}R^2 \\ &\leq Cr^2 - \frac{3}{2}R^2 < 0. \end{aligned}$$

Finally, case (iv) is impossible if  $0 < r \leq cR$ , since

$$\begin{aligned} 0 \leq \Gamma_u(y, t) \leq \varphi(y, t) = \varphi(y, -1) &\leq -R^2 + 1 + \frac{1}{2}A\beta(r + \varepsilon)^2 + \frac{1}{2}\sigma\beta^2(r + \varepsilon)^2 \\ &\leq -R^2 + Cr^2 < 0. \end{aligned}$$

This completes the proof of (4.3.27) and the lemma.  $\square$

## 4.4 Appendix A: continuous version of the doubling lemma

In this appendix, we provide a continuous version of the doubling variable maximum principle we used to compare the viscosity solution and the approximation solution.

**Proposition 4.4.1.** *Let  $\Omega \subseteq \mathbb{R}^{d+1}$ . Assume that  $u \in C^{0,\alpha}(\Omega)$  and  $v : \Omega \rightarrow \mathbb{R}$  satisfy*

$$\partial_t u + \mathcal{P}_{1,\Lambda}^-(D^2 u) \leq 1 \quad \text{in } \Omega \quad (4.4.1)$$

and

$$\partial_t v + \mathcal{P}_{1,\Lambda}^+(D^2 v) \geq -1 \quad \text{in } \Omega.$$

Fix  $0 < \delta \leq 1$  and define, for each  $(p, q, h, k) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ , a function  $\Phi : \Omega \times \Gamma_\varepsilon \rightarrow \mathbb{R}$  by

$$\Phi_{p,q,h,k}(x, t, y, s) := u(x, t) - v(y, s) - \frac{1}{2\delta} (|x - y|^2 + (t - s - h)^2) - p \cdot x - q \cdot y - k.$$

Assume that  $V, W \subseteq \mathbb{R}^{2d+2}$  and  $W \subseteq \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$  are such that  $\bar{V} \subseteq \Omega \times \Omega$  and, for each  $(p, q, h, k) \in W$ , there exists  $(x, t, y, s) \in V$  such that

$$0 = \Phi_{p,q,h,k}(x, t, y, s) = \sup \{ \Phi_{p,q,h,k}(\xi, \tau, \eta, \sigma) : (\xi, \tau, \eta, \sigma) \in \Omega \times \Omega, \tau \leq t \}$$

Then there exists a constant  $C > 1$  depending only on  $d, \Lambda, \text{diam } \Omega$  such that for all  $\rho > 0$ , the set  $\tilde{V}_\rho := V + Q_\rho^f \times Q_\rho^f$  satisfies

$$|W| \leq C \delta^{-2(d+1)} \left( 1 + \frac{\delta^{1/2}}{\rho} \right)^{2d+2} |\tilde{V}_\rho|.$$

with  $Q_\rho^f = B_\rho \times (0, \rho^2)$ .

**Proof of Proposition 4.4.1.** Suppose that, for each  $i \in \{1, 2\}$ ,

$$(x_i, t_i, y_i, s_i, p_i, q_i, h_i, k_i) \in \Omega \times \Gamma_\varepsilon \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$$

satisfies

$$\begin{aligned} 0 &= \Phi_{p_i, q_i, h_i, k_i}(x_i, t_i, y_i, s_i) \\ &= \sup \{ \Phi_{p_i, q_i, h_i, k_i}(\xi, \tau, \eta, \sigma) : (\xi, \tau, \eta, \sigma) \in \Omega \times \Gamma_\varepsilon, \tau \leq t_i \}. \end{aligned} \quad (4.4.2)$$

Denote

$$r := (|x_1 - x_2|^2 + |y_1 - y_2|^2 + |t_1 - t_2| + |s_1 - s_2|)^{1/2}.$$

The main step in the argument is to show that, for some  $C(d, \Lambda, K_0) > 1$ ,

$$|p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2 \leq C \delta^{-2} r^2. \quad (4.4.3)$$

For  $i \in \{1, 2\}$ , we introduce the intermediate quantities  $H_i$  and  $K_i$  at the touching points given by

$$\begin{aligned} H_i &:= u(x_i, t_i) - p_i \cdot x_i - \frac{1}{2\delta} (|x_i - y_i|^2 + (t_i - s_i - h_i)^2) \\ &= \sup_{(\xi, \tau) \in \Omega, \tau \leq t} \left( u(\xi, \tau) - p_i \cdot \xi - \frac{1}{2\delta} (|\xi - y_i|^2 + (\tau - s_i - h_i)^2) \right). \end{aligned} \quad (4.4.4)$$

and

$$\begin{aligned} K_i &:= -v(y_i, s_i) - \frac{1}{2\delta} (|x_i - y_i|^2 + (t_i - s_i - h_i)^2) - q_i \cdot y_i \\ &= \sup_{(\eta, \sigma) \in \Gamma_\varepsilon} \left( -v(\eta, \sigma) - q_i \cdot \eta - \frac{1}{2\delta} (|x_i - \eta|^2 + (t_i - \sigma - h_i)^2) \right). \end{aligned} \quad (4.4.5)$$

Notice that  $k_i$  can be expressed by

$$k_i = H_i + K_i + \frac{1}{2\delta} (|x_i - y_i|^2 + (t_i - s_i)^2) \quad (4.4.6)$$

The first three steps contain the proof of (4.3.2) and in the final step we deduce the proposition from (4.3.2). Throughout we let  $C$  and  $c$  denote positive constants which may vary from line to line and depend only on  $(d, \Lambda, K_0)$ .

We may assume without loss of generality that

$$t_1 - h_1 \leq t_2 - h_2. \quad (4.4.7)$$

*Step 1.* We claim that for each  $i \in \{1, 2\}$ ,

$$\max\{|x_i - y_i|^2, |t_i - s_i - h_i|^2\} \leq 4 \left( \max\{\|u\|_{L^\infty(\Omega)} + \text{diam } \Omega |p_i|, \|v\|_{L^\infty(\Omega)} + \text{diam } \Omega |q_i|\} \right) \delta. \quad (4.4.8)$$

We may assume without loss of generality that  $i = 1$ . We distinguish two cases. Assuming first that  $s_1 + h_1 \leq t_1$ , the definition (4.4.4) of  $H_1$  gives

$$u(x_1, t_1) - p_1 \cdot x_1 - \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1 - h_1)^2) \geq u(y_1, s_1 + h_1) - p_1 \cdot y_1,$$

which implies after rearranging the terms that

$$\begin{aligned} \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1 - h_1)^2) &\leq u(x_1, t_1) - u(y_1, s_1 + h_1) - p_1 \cdot (x_1 - y_1) \\ &\leq 2 \|u\|_{L^\infty(\Omega)} + 2|p_1| \text{diam } \Omega, \end{aligned}$$

which yields the result. Assume now that  $t_1 \leq s_1 + h_1$ , we get by definition of  $K_1$  that

$$-v(y_1, s_1) - \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1 - h_1)^2) - q_1 \cdot y_1 \geq -v(x_1, t_1 + h_1) - q_1 \cdot x_1.$$

This implies

$$\begin{aligned} \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1 - h_1)^2) &\leq v(x_1, t_1 + h_1) - v(y_1, s_1) + q_1 \cdot (x_1 - y_1) \\ &\leq 2 \|v\|_{L^\infty(\Omega)} + 2|q_1| \text{diam } \Omega, \end{aligned}$$

By putting together the two cases, we get (4.3.7).

*Step 2.* We claim that

$$|p_1 - p_2|^2 + |H_1 - H_2|^2 \leq C\delta^{-2}(1 + r^2)^3 r^2. \quad (4.4.9)$$

We are going to distinguish two cases. First assume that  $s_1 + h_1 \leq s_2 + h_2$ . Define

$$\tilde{u}(x, t) := -u(x, t) + \frac{1}{2\delta} (|x - y_1|^2 + (t - s_1 - h_1)^2).$$

Let  $T_m := \min\{s_1, t_1, t_2\}$  and  $T_M := \max\{s_2, t_1, t_2\}$ . Note that  $T_M - T_m \leq C + r$ . By (4.4.1) and Step 1, we find that, for  $\rho := C(1 + r)$ , we have  $(x_i, t_i), (y_i, s_i) \in Q_\rho(x_1, T_M)$  for each  $i \in \{1, 2\}$  and

$$\begin{aligned} \partial_t \tilde{u} + \mathcal{P}_{1,\Lambda}^+(D^2 \tilde{u}) &= -\partial_t u + \delta^{-1}(t - s_1 - h_1) + \mathcal{P}_{\lambda,\Lambda}^+ \left( -D^2 u + \frac{1}{\delta} I_d \right) \\ &\geq -C\delta^{-1} \quad \text{in } Q_\rho(x_1, T_M). \end{aligned} \quad (4.4.10)$$

According to (4.4.2) with  $i = 1$ , we have

$$\tilde{u}(x_1, t_1) = \Gamma_{\tilde{u}}(x_1, t_1) \quad \text{and} \quad (-p_1, \tilde{u}(x_1, t_1) + p_1 \cdot x_1) \in \partial \Gamma_{\tilde{u}}(x_1, t_1). \quad (4.4.11)$$

Note that  $H_1 = -(\tilde{u}(x_1, t_1) + p_1 \cdot x_1)$ . We next check that (4.4.2) with  $i = 2$  implies

$$\left( -p_2 + \frac{y_2 - y_1}{\delta}, \tilde{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \in \partial \Gamma_{\tilde{u}}(x_2, t_2). \quad (4.4.12)$$

To see this, note that the former implies that the map

$$(x, t) \mapsto u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x$$

achieves its supremum over  $\Omega \cap \{t \leq t_2\}$  at  $(x_2, t_2)$ , and we may write this as

$$\begin{aligned} u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x \\ = -\tilde{u}(x, t) - \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x - \frac{1}{\delta} (s_1 + h_1 - (s_2 + h_2)) t \\ + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 + (s_2 + h_2)^2). \end{aligned}$$

Using that  $s_1 + h_1 \leq s_2 + h_2$ , we obtain

$$\begin{aligned} \inf_{(x,t) \in \Omega, t \leq t_2} \left[ \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t \right] \\ \leq \inf_{(x,t) \in \Omega, t \leq t_2} \left[ \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t \right] \\ = \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 + \left(\frac{s_1 + h_1 - s_2 - h_2}{\delta}\right) t_2, \end{aligned}$$

that is,

$$(x, t) \mapsto \tilde{u}(x, t) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x \text{ attains its infimum on } \Omega \cap \{t \leq t_2\} \text{ at } (x_2, t_2).$$

This yields (4.4.12). Applying Lemma 4.3.1 in view of (4.4.16), (4.4.11) and (4.4.12) yields

$$\begin{aligned} \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + \left| \tilde{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 \right) \right|^2 \\ \leq C\delta^{-2} (|x_1 - x_2|^2 + |t_1 - t_2|^2), \quad (4.4.13) \end{aligned}$$

First, we have

$$|p_1 - p_2|^2 \leq 2 \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + 2\delta^{-2} |y_2 - y_1|^2 \leq C\delta^{-2} (|x_1 - x_2|^2 + |y_2 - y_1|^2 + |t_1 - t_2|^2).$$

It remains to estimate  $|H_1 - H_2|^2$ . By inserting the value of  $H_2$ , a direct computation gives

$$\begin{aligned} \tilde{u}(x_2, t_2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 \\ = -u(x_2, t_2) + \frac{1}{2\delta} (|x_2 - y_1|^2 + (t_2 - s_1 - h_1)^2) + \left(p_2 - \frac{y_2 - y_1}{\delta}\right) \cdot x_2 \\ = -H_2 + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 - (s_2 + h_2)^2 - 2(s_1 + h_1 - s_2 - h_2)t_2) \quad (4.4.14) \end{aligned}$$

We have

$$\begin{aligned} \left| |y_1|^2 - |y_2|^2 + (s_1 + h_1)^2 - (s_2 + h_2)^2 - 2(s_1 - s_2 + h_1 - h_2)t_2 \right|^2 \\ \leq 3 \left| |y_1|^2 - |y_2|^2 \right|^2 + 3 \left| (s_1 + h_1)^2 - (s_2 + h_2)^2 \right|^2 + 12 \left| (s_1 - s_2 + h_1 - h_2)t_2 \right|^2 \\ \leq C(\text{diam } \Omega)^2 |y_1 - y_2|^2 + \left( |s_1 + h_1 + s_2 + h_2|^2 + 12t_2^2 \right) |(s_1 + h_1) - (s_2 + h_2)|^2 \\ \leq C(\text{diam } \Omega)^2 |y_1 - y_2|^2 + C(\max\{h_1^2, h_2^2\} + T^2) ((t_1 - t_2)^2 + \delta) \\ \leq C(\text{diam } \Omega + \max\{|h_1|, |h_2|\} + T)^2 (\delta + |t_1 - t_2|^2 + |y_1 - y_2|^2). \quad (4.4.15) \end{aligned}$$

where in the third line, it was Step 1 that gave us

$$|(s_1 + h_1) - (s_2 + h_2)| \leq |t_1 - t_2| + C\delta^{1/2}.$$

Thanks to (4.4.11), (4.4.14), (4.4.15) and (4.4.13), we deduce that

$$\begin{aligned} |H_1 - H_2|^2 &\leq 2 \left| \tilde{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \tilde{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \right|^2 + 2L^2 \\ &\leq C(1 + \text{diam } \Omega + \max\{|h_1|, |h_2|\} + T)^2 \delta^{-2} (\delta + |x_1 - x_2|^2 + |t_1 - t_2|^2 + |y_1 - y_2|^2). \end{aligned}$$

• Case 2:  $s_1 + h_1 > s_2 + h_2$ . Define

$$\hat{u}(x, t) := -u(x, t) + \frac{1}{2\delta} (|x - y_1|^2 + (t - s_1 - h_1)^2) + \left( \frac{s_1 + h_1 - s_2 - h_2}{\delta} \right) t.$$

We compute

$$\begin{aligned} \partial_t \hat{u} + \mathcal{P}_{1,\Lambda}^+(D^2 \hat{u}) &= -\partial_t u + \delta^{-1}(t - s_2 - h_2) + \mathcal{P}_{\lambda,\Lambda}^+ \left( -D^2 u + \frac{1}{\delta} I_d \right) \\ &\geq -C\delta^{-1} \quad \text{in } Q_\rho(x_1, T_M). \end{aligned} \quad (4.4.16)$$

Using (4.4.11) and  $s_1 + h_1 > s_2 + h_2$ , we deduce that

$$\hat{u}(x_1, t_1) = \Gamma_{\hat{u}}(x_1, t_1) \quad \text{and} \quad (-p_1, \hat{u}(x_1, t_1) + p_1 \cdot x_1) \in \partial \Gamma_{\hat{u}}(x_1, t_1). \quad (4.4.17)$$

We get that

$$\begin{aligned} u(x, t) - \frac{1}{2\delta} (|x - y_2|^2 + (t - s_2 - h_2)^2) - p_2 \cdot x \\ = -\hat{u}(x, t) - \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x + \frac{1}{2\delta} (|y_1|^2 + (s_1 + h_1)^2 - |y_2|^2 + (s_2 + h_2)^2), \end{aligned}$$

that is,

$$(x, t) \mapsto \hat{u}(x, t) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x \text{ attains its infimum on } \Omega \cap \{t \leq t_2\} \text{ at } (x_2, t_2).$$

This yields (4.4.12). Applying Lemma 4.3.1 in view of (4.4.16), (4.4.11) and (4.4.12) yields

$$\begin{aligned} \left| p_1 - p_2 + \frac{y_2 - y_1}{\delta} \right|^2 + \left| \hat{u}(x_1, t_1) + p_1 \cdot x_1 - \left( \hat{u}(x_2, t_2) + \left( p_2 - \frac{y_2 - y_1}{\delta} \right) \cdot x_2 \right) \right|^2 \\ \leq C\delta^{-2} (|x_1 - x_2|^2 + |t_1 - t_2|^2). \end{aligned}$$

*Step 4.(Conclusion)* First we show that

$$|k_1 - k_2|^2 \leq C\delta^{-2}(\delta + (1 + r^2)r^2).$$

since by (4.3.5), we can write

$$|\hat{k}_1 - \hat{k}_2| \leq |H_1 - H_2| + |K_1 - K_2| + \frac{1}{2\delta} (|x_1 - y_1|^2 + (t_1 - s_1)^2 - |x_2 - y_2|^2 - (t_2 - s_2)^2)$$

and the claim directly follows from Steps 2-3 and the assumption  $|k_i - \hat{k}_i| \leq (C\varepsilon^\alpha + \varepsilon^2\delta^{-1/2})$ . By combining Steps 2-5, we obtain

$$\begin{aligned} |p_1 - p_2|^2 + |q_1 - q_2|^2 + |h_1 - h_2|^2 + |k_1 - k_2|^2 &\leq C\delta^{-2}(\delta + (1 + r^2)r^2) \\ &\leq C\delta^{-1}(\delta^{1/2} + (1 + r^2)^{1/2}r). \end{aligned}$$

Since  $r \leq \text{diam } \Omega$ , there is a constant  $C$  depending on  $T$  and  $\text{diam } \Omega$  such that

$$(|p_1 - p_2|^2 + |h_1 - h_2|^2 + |a_1 - a_2|^2 + |b_1 - b_2|^2)^{1/2} \leq C\delta^{-1}(\delta^{1/2} + r).$$

By applying Lemma 4.3.3, we deduce that

$$|W| \leq C\delta^{-(2d+2)} \left(1 + \frac{\delta^{1/2}}{\rho}\right)^{2d+2} |\tilde{V}_\rho|.$$

which yields the desired inequality.  $\square$

## 4.5 Appendix B: Proof of Proposition 4.2.5

This section is devoted to the proof of Proposition 4.2.5 on the rate outside the cylinder  $\tilde{Q} = B_{R_1} \times [0, T]$ . First we are going to study the rate of convergence for small times which consists to study the consistency of the approximation solution with the initial datum  $g$ . Then we will give the rate far from the origin in space.

**Proposition 4.5.1** (Rate for small times). *Let  $\mathcal{F}$  be uniformly elliptic,  $g \in C^{0,1}(\mathbb{R}^d)$  and  $v$  be the solution of (4.1.3). Then there exists a universal constant  $C$  such that, for all  $x_0 \in \mathbb{R}^d$  and  $(z, s) \in \Gamma_\varepsilon$ , we have*

$$|g(x_0) - v(z, s)| \leq \|g\|_{C^{0,1}} R_0 \varepsilon + C(1 + \|g\|_{C^{0,1}}) \text{dist}_p((z, s)(x_0, 0)).$$

*Proof.* Define

$$\phi(x, t) := g(x_0) - \|g\|_{C^{0,1}} R_0 \varepsilon - \left(1 + \Lambda d \frac{\|g\|_{C^{0,1}}}{R_0 \varepsilon}\right) t - \frac{\|g\|_{C^{0,1}}}{R_0 \varepsilon} |x - x_0|^2.$$

We are going to show that  $\phi(\cdot, 0) \leq g$ , the proof of the upper inequality following exactly the same lines. We distinguish two cases. First assume that  $|x - x_0| \leq R_0 \varepsilon$ , then

$$\phi(x, 0) \leq g(x_0) - \|g\|_{C^{0,1}} R_0 \varepsilon \leq g(x - x_0)$$

where we used for the last inequality that  $g$  is Lipschitz. Suppose now that  $|x - x_0| > R_0 \varepsilon$ , then

$$\begin{aligned} \phi(x, 0) &= g(x_0) - \|g\|_{C^{0,1}} R_0 \varepsilon - \left(\frac{\|g\|_{C^{0,1}}}{R_0 \varepsilon} |x - x_0|\right) |x - x_0| \\ &\leq g(x_0) - \|g\|_{C^{0,1}} R_0 \varepsilon - \|g\|_{C^{0,1}} |x - x_0| \leq g(x). \end{aligned}$$

We compute

$$\hat{\partial}_t^\varepsilon \phi + F_\varepsilon[\phi] = \partial_t \phi + \mathcal{F}(D^2 \phi) \leq \partial_t \phi + \mathcal{P}_{\lambda, \Lambda}^+(D^2 \phi) = -1.$$

Since  $v$  is a supersolution of the equation above, applying Proposition 4.2.2 yields  $v \geq \phi$ .  $\square$

*Proof of Proposition 4.2.5, (4.2.5).* We are going to study the case  $|x| \geq R_1$ , the inequality on time being strictly parallel. The idea is to compare  $u$  and  $v$  to some suitable gaussians. This procedure is fine since the viscosity solution  $u$  of (4.1.1) satisfies

$$\partial_t u + \mathcal{P}_{\lambda, \Lambda}^-(D^2 u) \leq 0 \leq \partial_t u + \mathcal{P}_{\lambda, \Lambda}^+(D^2 u).$$

The null function is a subsolution of (4.1.1). By the maximum principle, we deduce that the viscosity solution  $u$  of (4.1.1) is nonnegative.

By combining Lemma 4.2.4 for the choice  $\tau_1 = \frac{1}{2B\lambda}$  and  $a_1 = (8\Lambda)^{-1}$ , we deduce that there exists a constant  $C_1 > 0$  such that

$$u(x, t) \leq C_1(t + \tau_1)^{-\alpha^+} \exp(-a_1|x|^2/(t + \tau_1)) \quad \text{for all } (x, t) \in \mathbb{R}^d \times \mathbb{R}_+.$$

Define the radius  $R_1 > 0$  such that

$$R_1 := \left| \frac{\tau_1}{\alpha} \ln(\tfrac{1}{2} C_1 \varepsilon^2 \tau^\alpha) \right|^{1/2}$$

Then we deduce from the inequality above that for all  $|x| \geq R_1$  and all  $t > 0$ ,

$$0 \leq u(x, t) \leq \varepsilon^2.$$

Then by developing a barrier argument similar to the proof of Proposition 4.5.1 and using that the gaussian  $\Phi$ , we get the same inequality on the approximate solution  $v$  which implies the desired bound.  $\square$





# Annexes



## Annexe A

# Estimation $W^{3,\varepsilon}$ elliptique

Dans cette courte annexe, nous donnons une preuve de l'estimation  $W^{3,\varepsilon}$  elliptique. Ce résultat, obtenu tout d'abord par Caffarelli, est bien connu mais il nous a paru pertinent de présenter ici cette preuve car elle met en oeuvre dans le cadre elliptique les outils strictement analogues à ceux mis en oeuvre dans le cas parabolique au Chapitre 2.

## A.1 Introduction

This short appendix is devoted to an elementary proof of the elliptic  $W^{3,\varepsilon}$  estimate by using the same technics used in the parabolic framework (see Chapter 2). We prove it for viscosity solutions of the uniformly elliptic equation

$$\mathcal{F}(D^2u) = g \quad \text{in } U \subseteq \mathbb{R}^d, \quad (\text{A.1.1})$$

where  $g \in C^{0,1}(U)$ . To give the precise statement of this result, we require some notation. We denote by  $\mathbb{M}_d$  the set of real  $d \times d$  matrices. The open ball of  $\mathbb{R}^d$  centered at  $x$  of radius  $\rho$  is denoted by  $B_\rho(x)$ . If  $x = 0$ , we simply write  $B_\rho$ .

Now we define, for  $u : U \rightarrow \mathbb{R}$ , the quantity

$$\Psi(u, U)(x) := \inf \left\{ A \geq 0 : \exists (p, M) \in \mathbb{R}^d \times \mathbb{M}_d \text{ s.t. } \forall y \in U, \right. \\ \left. |u(y) - u(x) - p \cdot (y - x) - \frac{1}{2}(y - x) \cdot M(y - x)| \leq \frac{1}{6}A|x - y|^3 \right\}.$$

We emphasize here that  $\Psi(u, U)(x)$  is defined in terms of cubic polynomials that touch  $u$  at  $x$  and stay below or above  $u$  in the whole domain  $U$ . The statement of the elliptic  $W^{3,\varepsilon}$  estimate is given by the following theorem.

**Theorem A.1.1** (Elliptic  $W^{3,\varepsilon}$  estimate). *Assume  $\mathcal{F}$  satisfies (F1),  $g \in C^{0,1}(B_1)$  and  $u \in C(B_1)$  solves (A.1.1) in  $B_1$ . Then there exist universal constants  $C, \varepsilon > 0$  such that, for all  $\kappa > 0$ ,*

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{B_1} |u| + |\mathcal{F}(0)| + \|g\|_{C^{0,1}(B_1)}} \right)^{-\varepsilon}. \quad (\text{A.1.2})$$

## A.2 Proof of Theorem A.1.1

To prove the elliptic  $W^{3,\varepsilon}$  estimate, we need to combine the elliptic  $W^{2,\varepsilon}$  estimate on the gradient functions  $u_e = Du \cdot e$ ,  $|e| = 1$ , and a lemma giving the link between  $\Psi$  and  $\Theta$ .

To state the  $W^{2,\varepsilon}$  estimate, we require some notation. Given a domain  $U \subseteq \mathbb{R}^d$ , and a function  $u \in \text{LSC}(U)$ , define the quantity

$$\underline{\Theta}(x) = \underline{\Theta}(u, U)(x) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ s.t. } \forall y \in U, u(y) \geq u(x) + p \cdot (y - x) - \frac{A}{2}|x - y|^2 \right\}.$$

Similarly, for  $u \in \text{USC}(U)$ ,

$$\overline{\Theta}(x) = \overline{\Theta}(u, U)(x) := \inf \left\{ A \geq 0 : \exists p \in \mathbb{R}^d \text{ s.t. } \forall y \in U, u(y) \leq u(x) + p \cdot (y - x) + \frac{A}{2}|x - y|^2 \right\},$$

and, for  $u \in C(U)$ ,

$$\Theta(x) = \Theta(u, U)(x) := \max \{ \underline{\Theta}(u, U)(x), \overline{\Theta}(u, U)(x) \}.$$

The quantity  $\underline{\Theta}(x, t)$  is the minimum curvature of any paraboloid that touches  $u$  from below at  $x$ . If  $u$  cannot be touched from below at  $x$  by any paraboloid, then  $\underline{\Theta}(x, t) = +\infty$ . A similar statement holds for  $\overline{\Theta}(x)$ , where we touch from above instead. Moreover, a function  $u$  is  $C^{1,1}$  on a closed set  $\Gamma \subseteq U$  if and only if  $u$  has tangent paraboloids from above and below with respect to  $U$  at each point of  $\Gamma$ .

The form of the  $W^{2,\varepsilon}$  estimate we need is given by the following proposition.

**Proposition A.2.1.** *If  $u \in \text{LSC}(Q_1)$  and  $L \geq 0$  satisfy the inequality*

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2u) \geq -L \quad \text{in } B_1,$$

*then for all  $\kappa > 0$ ,*

$$|\{x \in B_{1/2} : \underline{\Theta}(u, B_1)(x) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{B_1} |u| + L} \right)^{-\varepsilon}, \quad (\text{A.2.1})$$

*where the constants  $C$  and  $\varepsilon > 0$  are universal.*

**Lemma A.2.2** (Armstrong, Smart, Silvestre [2]). *Assume that  $u \in C^1(B_1)$ . Then for each  $x \in B_1$ ,*

$$\Psi(u, B_1)(x) \leq \left( \sum_{i=1}^d (\Theta(u_{x_i}, B_1)(x))^2 \right)^{1/2}.$$

*Proof of Theorem A.1.1.* If  $u \equiv 0$  on  $Q_1$ , then the estimate is clear, so we may assume that  $\sup_{Q_1} |u| > 0$ . In Step 1, we reduce the proof of the theorem to the case of  $\sup_{Q_1} |u| \leq 1$ ,  $g(0) = 0$  and  $\mathcal{F}(0) = 0$  by scaling arguments. In Step 2, we prove the theorem under these assumptions.

*Step 1.* We first reduce to the case that  $\mathcal{F}(0) = 0$ . If  $\mathcal{F}(0) \neq 0$ , then, by ellipticity, there exists  $a \in \mathbb{R}$  such that

$$\mathcal{F}(aI_d) = 0 \quad \text{with } |a| \leq \frac{1}{\lambda d} |\mathcal{F}(0)|. \quad (\text{A.2.2})$$

Define the operator  $\widehat{\mathcal{F}}(M) := \mathcal{F}(M + aI_d)$  and observe that  $\widehat{\mathcal{F}}$  satisfies (F1) with the same ellipticity constants  $\lambda, \Lambda$  and  $\widehat{\mathcal{F}}(0) = \mathcal{F}(aI_d) = 0$ . It is clear that  $\widehat{u}(x) := u(x) - \frac{1}{2}a|x|^2$  is a solution of

$$\widehat{\mathcal{F}}(D^2\widehat{u}) = g.$$

By direct computation,

$$\sup_{Q_1} |\widehat{u}| \leq \sup_{Q_1} |u| + \frac{1}{2}|a| \sup_{x \in B_1} |x|^2 \leq \sup_{Q_1} |u| + \frac{1}{2}|a|.$$

By applying (A.2.4) and using (A.2.2), we obtain

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{B_1} |u| + \frac{1}{2\lambda d} |\mathcal{F}(0)|} \right)^{-\varepsilon},$$

and we get the inequality given by Theorem A.1.1.

Next we reduce to the case that  $\sup_{B_1} |u| \leq 1$ . Assume that we have shown if  $\sup_{B_1} |u| \leq 1$  and  $\mathcal{F}(0) = 0$ , then for all  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(B_1)}} \right)^{-\varepsilon}. \quad (\text{A.2.3})$$

We claim that if  $\mathcal{F}(0) = 0$ , and  $\beta := \sup_{B_1} |u| > 0$  then for all  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{\beta + \|g\|_{C^{0,1}(B_1)}} \right)^{-\varepsilon}. \quad (\text{A.2.4})$$

Define the function  $\tilde{g} := (1/\beta)g$ , the operator  $\widetilde{\mathcal{F}}(M) := \beta^{-1}\mathcal{F}(\beta M)$  and observe  $\tilde{g} \in C^{0,1}(B_1)$  with  $\|\tilde{g}\|_{C^{0,1}(B_1)} = \|g\|_{C^{0,1}(B_1)} / \beta$ ,  $\widetilde{\mathcal{F}}$  satisfies (F1) with the same ellipticity constants  $\lambda, \Lambda$  and  $\widetilde{\mathcal{F}}(0) = \beta^{-1}\mathcal{F}(0) = 0$ . It is clear that  $\tilde{u} := u/\beta$  is a solution of

$$\widetilde{\mathcal{F}}(D^2\tilde{u}) = \tilde{g}.$$

with  $\sup_{B_1} |\tilde{u}| = 1$ . By applying (A.2.3) to  $\tilde{u}$ , we obtain that, for all  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \Psi(\tilde{u}, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(B_1)} / \beta} \right)^{-\varepsilon}.$$

Noticing that  $\Psi(\tilde{u}, B_{3/4})(x) = \frac{1}{\beta} \Psi(u, B_{3/4})(x)$ , we obtain that, for all  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \beta\kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(B_1)} / \beta} \right)^{-\varepsilon}.$$

This is equivalent to (A.2.4).

*Step 2.* Assuming that  $\sup_{B_1} |u| \leq 1$ ,  $g(0) = 0$  and  $\mathcal{F}(0) = 0$ , we give the proof of (A.2.3). It suffices to get the inequality for  $\kappa \geq \kappa_1$ , where  $\kappa_1$  is a universal constant. Since  $u$  satisfies an interior  $C^{1,\alpha}$  estimate,  $Du$  is continuous and there exists a constant  $C$  such that

$$\sup_{B_{3/4}} |Du| \leq C \left( 1 + \|g\|_{C^{0,1}(B_1)} \right). \quad (\text{A.2.5})$$

By [34, Lemma 3.12], we know that, for every unit direction  $e \in \mathbb{R}^d$ ,  $|e| = 1$ , the function  $u_e = e \cdot Du$  satisfies the inequalities

$$\mathcal{P}_{\lambda,\Lambda}^-(u_e) - \|g\|_{C^{0,1}(B_1)} \leq 0 \leq \mathcal{P}_{\lambda,\Lambda}^+(u_e) + \|g\|_{C^{0,1}(B_1)}, \quad \text{in } B_1,$$

in the viscosity sense. According to Proposition A.2.1, we have, for each  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \underline{\Theta}(u_e, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{\sup_{B_{3/4}} |u_e| + \|g\|_{C^{0,1}(B_1)}} \right)^{-\varepsilon},$$

where  $C, \varepsilon > 0$  are universal constants. Thus, we deduce from (A.2.5) that there exists a new constant  $C > 0$  such that for all  $\kappa > 0$ ,

$$|\{x \in B_{1/2} : \underline{\Theta}(u_e, B_{3/4})(x) > \kappa\}| \leq C \left( \frac{\kappa}{1 + \|g\|_{C^{0,1}(B_1)}} \right)^{-\varepsilon}. \quad (\text{A.2.6})$$

By Lemma A.2.2, we have for all  $\kappa > 0$  that

$$|\{(x \in B_{1/2} : \Psi(u, B_{3/4})(x) > \kappa)\}| \leq \sum_{i=1}^d \left| \left\{ x \in B_{1/2} : \Theta(u_{x_i}, B_{3/4})(x) > \frac{\kappa}{\sqrt{d}} \right\} \right|,$$

and we obtain the desired result by applying (A.2.6). This completes the proof of (A.2.3).  $\square$

### A.3 Proof of Proposition A.2.1

Instead of working with the sets  $\{\underline{\Theta} \leq \kappa\}$ , we are going to consider some new sets  $A_\kappa$  for  $\kappa > 0$ . Here we give the elliptic definition introduced by Savin [126] and recently also used by Armstrong and Smart in [5]. Define, for every  $\kappa > 0$ ,

$$A_\kappa := \left\{ x \in B_1 : \exists y \in B_1 \text{ such that } u(x) + \frac{\kappa}{2}|x - y|^2 = \inf_{z \in B_1} \left( u(z) + \frac{\kappa}{2}|z - y|^2 \right) \right\}. \quad (\text{A.3.1})$$

Moreover, the definition of  $A_\kappa$  given is adapted to the domain  $B_1$ . It is clear how to change this definition of  $A_\kappa$  to deal with more general domains.

The next lemma gather some properties about the sets  $A_\kappa$ . In particular, the link between  $A_\kappa$  and  $\underline{\Theta}$  is precised.

**Lemma A.3.1.** *Let  $u \in \text{LSC}(\Omega)$ ,  $\kappa > 0$  and  $A_\kappa$  be defined by (A.3.1). Then we have*

$$A_\kappa \subseteq \{x \in B_1 : \underline{\Theta}(u, B_1) \leq \kappa\}. \quad (\text{A.3.2})$$

Moreover, for all  $0 \leq \kappa_1 \leq \kappa$ , we have  $A_{\kappa_1} \subseteq A_\kappa$ .

The following lemma is the form of the ABP inequality we are going to use.

**Lemma A.3.2.** Assume that  $L > 0$  and  $u \in \text{LSC}(B_1)$  satisfy

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2 u) \geq -L \text{ in } B_1.$$

Suppose that  $a > 0$  and  $V \subseteq \mathbb{R}^d$  is compact such that, for each  $(y, s) \in V$ , there exists  $(x, t) \in B_1$  such that

$$u(x) + \frac{a}{2}|x - y|^2 = \inf_{z \in B_1} (u(z) + \frac{a}{2}|z - y|^2). \quad (\text{A.3.3})$$

Let  $W := \{x \in B_1 : (\text{A.3.3}) \text{ holds for } u \text{ for some } y \in V\}$ . Then

$$|V| \leq \frac{1}{\lambda^d} \left( \frac{L}{a} + \Lambda d \right)^d |W|. \quad (\text{A.3.4})$$

*Proof.* The proof is divided into two steps.

**Step 1.** We make two reductions. First, by replacing  $u$  by  $u + \frac{\alpha}{2}|x|^2$  and  $L$  by  $L + C\alpha$  and letting  $\alpha \rightarrow 0$ , we may suppose that there exists  $\eta > 0$  such that for every  $y \in B_1$ ,

$$\min_{z \in B_1 \setminus B_{1-\eta}} (u(z) + \frac{a}{2}|z - y|^2) > \inf_{z \in B_{1-\eta}} (u(z) + \frac{a}{2}|z - y|^2). \quad (\text{A.3.5})$$

Next we make a reduction to the case that  $u$  is semiconcave by an infimal convolution approximation. According to (A.3.5), for every sufficiently small  $\varepsilon > 0$ , there exists  $0 < r_\varepsilon < 1$  such that, for each  $y \in V$ , there exists  $x \in B_{1-r_\varepsilon}$  such that

$$u_\varepsilon(x) + \frac{a}{2}|x - y|^2 = \inf_{z \in B_{1-r_\varepsilon}(0)} (u(z) + \frac{a}{2}|z - y|^2), \quad (\text{A.3.6})$$

and

$$\lim_{\varepsilon \rightarrow 0} r_\varepsilon = 0.$$

Set

$$W_\varepsilon := \{x \in B_{1-r_\varepsilon} : (\text{A.3.6}) \text{ holds for } u_\varepsilon \text{ for some } y \in V\}.$$

Assume that we have shown that

$$\limsup_{\varepsilon \rightarrow 0} W_\varepsilon \subseteq W, \quad \text{where} \quad \limsup_{\varepsilon \rightarrow 0} W_\varepsilon := \bigcap_{\varepsilon > 0} \bigcup_{0 < \delta \leq \varepsilon} W_\delta, \quad (\text{A.3.7})$$

and for all  $\varepsilon > 0$ ,

$$|V| \leq \frac{1}{\lambda^d} \left( \frac{L}{a} + \Lambda d \right)^{d+1} |W_\varepsilon|. \quad (\text{A.3.8})$$

Then, since  $\sup_{0 < \varepsilon < 1/2} |W_\varepsilon| \leq |Q_1| < +\infty$ , the inequality  $\limsup_{\varepsilon \rightarrow 0} |W_\varepsilon| \leq |\limsup_{\varepsilon \rightarrow 0} W_\varepsilon|$  holds true and we have

$$|V| \stackrel{(\text{A.3.8})}{\leq} \frac{1}{\lambda^d} \left( \frac{L}{a} + \Lambda d \right)^{d+1} \left| \limsup_{\varepsilon \rightarrow 0} W_\varepsilon \right| \stackrel{(\text{A.3.7})}{\leq} \frac{1}{\lambda^d} \left( \frac{L}{a} + \Lambda d \right)^{d+1} |W|.$$

Thus we deduce (A.3.4).

To obtain the lemma, it remains to show the assertions (A.3.7) and (A.3.8).

For (A.3.7), let  $x \in \limsup_{\varepsilon \rightarrow 0} W_\varepsilon$ . Up to a subsequence, we can assume that  $x \in W_\varepsilon \cap B_{1-r_\varepsilon}$  for all  $\varepsilon > 0$ . By the definition of  $W_\varepsilon$ , there exists  $y_\varepsilon \in V$  such that

$$u_\varepsilon(x) + \frac{a}{2}|x - y_\varepsilon|^2 = \inf_{z \in B_{1-r_\varepsilon}} [u_\varepsilon(z) + \frac{a}{2}|z - y_\varepsilon|^2]. \quad (\text{A.3.9})$$

Since  $V$  is compact, up to extracting a subsequence, there exists  $y \in V$  such that  $y_\varepsilon \rightarrow y$  as  $\varepsilon \rightarrow 0$ . By convergence of  $u_\varepsilon$ , we deduce that  $u_\varepsilon(x) \xrightarrow{\varepsilon \rightarrow 0} u(x)$ . Letting  $\varepsilon \rightarrow 0$  in (A.3.9) yields  $x \in W$ . This completes the proof of (A.3.7), and therefore it remains to prove (A.3.8), that is, the statement of the lemma under the extra assumption that  $u$  is semiconcave.



**Step 2.** Assuming  $u$  is semiconcave, we give the proof of (A.3.4). Select a Lebesgue-measurable function  $\bar{Z} : V \rightarrow B_1$  such that the map

$$z \mapsto u(z) - \inf_{B_1} u + a \left( \frac{1}{2} |z - y|^2 \right)$$

attains its infimum in  $B_1$  at  $z = \bar{Z}(y)$  and this infimum is equal to zero. For example, we may take  $\bar{Z}(y)$  to be the lexicographically least element of the (necessarily closed) set of infima. The function  $u$  is  $C^{1,1}$  on  $A := \bar{Z}(V)$  and  $\bar{Z}$  has a Lipschitz inverse  $\bar{Y}$  given by

$$\bar{y}(z) = z + \frac{1}{a} Du(z).$$

By Rademacher's theorem,  $\bar{Y}$  is differentiable almost everywhere on  $A$  for the  $d$ -dimensional Lebesgue measure. Then, by using the Lebesgue differentiation theorem, we see that  $u$  is twice differentiable in space at almost every point of  $z \in A$  and, at such  $z$ , we have

$$D^2 u(z) \geq -a I_d.$$

Thus,

$$D\bar{y}(z) = I_d + \frac{1}{a} D^2 u(z) \geq 0$$

as well as

$$-\lambda \operatorname{tr}(D\bar{y}(z)) = \mathcal{P}_{\lambda,\Lambda}^+(D\bar{y}(z)) = \mathcal{P}_{\lambda,\Lambda}^+ \left( I_d + \frac{1}{a} D^2 u(z) \right) \geq \frac{1}{a} \mathcal{P}_{\lambda,\Lambda}^+(D^2 u(z)) + \mathcal{P}_{\lambda,\Lambda}^-(I_d) \geq -\frac{L}{a} - \Lambda d,$$

and therefore

$$0 \leq D\bar{y}(z) \leq \frac{1}{\lambda} \left( \frac{L}{a} + \Lambda d \right).$$

An application of the area formula for Lipschitz function gives

$$|V| = \int_A |\det D\bar{Y}(z)| dz = \int_A \det \left( I + \frac{1}{a} D^2 u(z) \right) dz \leq \frac{1}{\lambda^d} \left( \frac{L}{a} + \Lambda d \right)^d |A|,$$

from which we obtain the lemma, using that  $A \subseteq W$ .  $\square$

In our analysis, an important role will be played by the functions  $\phi$  which we define by

$$\phi(x, t) := c \left( |x|^{-a} - r^{-a} \right).$$

The parameters  $a, b$  and  $c$  will be adjusted with the uniform ellipticity constants and the opening  $\theta$  of the parabolic balls of the form  $G_{\theta, 1+\tau}(0, -\tau)$  with  $\tau > 0$ . More precisely, we will consider the choice given by

$$a = \max \left\{ \frac{(d-1)\Lambda + 1}{\lambda}, 1 \right\} \quad (\text{A.3.10})$$

and

$$c = \max \{1, r^{a+2}\}. \quad (\text{A.3.11})$$

We next show that, with this choice of parameters,  $\phi$  is a nonnegative subsolution in  $B_r \setminus B_{r/8}$  which vanishes on the lateral boundary of  $B_r$  and is not too large initially. This plays the role of the “bump function” from the elliptic case [34, Lemma 4.1].

**Lemma A.3.3.** *For  $a$  and  $c$  given by (A.3.10)–(A.3.11), the function  $\phi$  satisfies*

$$\begin{cases} \mathcal{P}_{\lambda,\Lambda}^+(D^2 \phi) \leq -1, & \text{on } B_r \setminus B_{r/8}, \\ \phi = 0, & \text{on } \partial B_r, \\ \phi = \beta, & \text{on } \partial B_{r/8} \end{cases}$$

with  $\beta > 0$  given by

$$\beta := c(8^a - 1) r^{-a}.$$

*Proof.* The Hessian of  $\phi$  is given by

$$D^2\phi(x, t) = ca|x|^{-a-2} \left( (a+1) \frac{x \otimes x}{|x|^2} - \left( I - \frac{x \otimes x}{|x|^2} \right) \right).$$

and has eigenvalues  $(a+1)ca|x|^{-a-2}$  with multiplicity 1 and  $-ca|x|^{-a-2}$  with multiplicity  $d-1$ . Hence

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2\phi(x, t)) = ca|x|^{-a-2} ((d-1)\Lambda - (a+1)\lambda).$$

By inserting the value (A.3.10)–(A.3.11) of  $a$  and  $c$ , we check that

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2\phi(x, t)) = ca|x|^{-a-2} ((d-1)\Lambda - (a+1)\lambda) \leq -car^{-a-2} \leq -cr^{-a-2} \leq -1,$$

which yields the desired upper bound.  $\square$

The following lemma contains the measure theoretic information necessary to conclude the proof of Proposition A.2.1. The argument relies on Lemmas A.3.2 and A.3.3.

**Lemma A.3.4.** *Let  $r_0 > 0$ ,  $\kappa_1 > 0$ ,  $A_\kappa$  be defined by (A.3.1) and  $B_{r_0}(x_0) \subseteq B_1$ . There exist universal constants  $M \geq 1$  and  $\sigma > 0$  such that, if  $x_0 \in A_{\kappa_1}$  and  $\kappa \geq \kappa_1$ , then*

$$|A_{M\kappa} \cap B_{r_0}(x_0)| \geq |B_{r_0}(x_0) \cap A_\kappa| + \sigma\eta |B_{r_0}(x_0) \setminus A_\kappa|.$$

*Proof.* We decompose the measure estimate into two parts

$$|A_{M\kappa} \cap B_{r_0}(x_0)| = |A_\kappa \cap B_{r_0}(x_0)| + |(A_{M\kappa} \setminus A_\kappa) \cap B_{r_0}(x_0)|.$$

It is enough to estimate  $|(A_{M\kappa} \setminus A_\kappa) \cap B_{r_0}(x_0)|$ . We claim that

$$|B_{r_0}(x_0) \setminus A_\kappa| \leq \frac{1}{\sigma\eta} |(A_{M\kappa} \setminus A_\kappa) \cap B_{r_0}(x_0)|.$$

For  $\kappa \geq \kappa_1$ , we define the collection of parabolic balls given by

$$\mathcal{B} := \left\{ B_r(x) : x \in B_{r_0}(x_0), B_r(x) \cap B_{r_0}(x_0) \subseteq B_1 \setminus A_\kappa \text{ and } \partial B_r(x) \cap B_{r_0}(x_0) \cap A_\kappa \neq \emptyset \right\}.$$

Notice that for all  $x \in B_{r_0}(x_0)$ , the point  $x$  belongs to the ball  $\overline{B}_{r_0}(x)$ . Observe that  $x_0 \in A_\kappa$  since  $x_0 \in A_{\kappa_1}$ . This implies that for all  $B_r(x) \in \mathcal{B}$ ,  $r \leq r_0$ . Then, by Vitali's lemma, we may extract from  $\mathcal{B}$  a countable subcollection  $\{B_{r_i}(x_i) : i \in \mathbb{N}\}$  such that the  $B_{r_i}(x_i)$  are disjoint,

$$B_{r_0}(x_0) \setminus A_\kappa \subseteq \bigcup_{i \in \mathbb{N}} \widehat{B}_{r_i}(x_i) \quad \text{and} \quad \frac{|B_{r_i}(x_i)|}{|\widehat{B}_{r_i}(x_i)|} = \eta.$$

By combining these, we get

$$|B_{r_0}(x_0) \setminus A_\kappa| \leq \sum_{i \in \mathbb{N}} |\widehat{B}_{r_i}(x_i)| = \frac{1}{\eta} \sum_{i \in \mathbb{N}} |B_{r_i}(x_i)|.$$

Next we complete the proof under the assumption that

$$|B_{r_i}(x_i) \cap B_{r_0}(x_0) \cap A_{M\kappa}| \geq \sigma |B_{r_i}(x_i)| \tag{A.3.12}$$

for some constants  $M > 1$  and  $\sigma > 0$ , depending only on  $\theta$ ,  $d$ ,  $\lambda$  and  $\Lambda$ . Using also that the selected balls are disjoint, we obtain that

$$\sum_{i \in \mathbb{N}} |B_{r_i}(x_i)| \leq \frac{1}{\sigma} \sum_{i \in \mathbb{N}} |B_{r_i}(x_i) \cap B_{r_0}(x_0) \cap A_{M\kappa}| = \frac{1}{\sigma} \left| \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i) \cap B_{r_0}(x_0) \cap A_{M\kappa} \right|.$$

Since every ball in  $\mathcal{B}$  satisfies  $B_{r_i}(x_i) \cap B_{r_0}(x_0) \subseteq B_{r_0}(x_0) \setminus A_\kappa$ , we deduce that

$$|B_{r_0}(x_0) \setminus A_\kappa| \leq \frac{1}{\sigma\eta} |B_{r_0}(x_0) \cap (A_{M\kappa} \setminus A_\kappa)|.$$

The proof is complete, pending the verification of (A.3.12), which is achieved in the next lemma.  $\square$

The following lemma is the key step in the proof of Proposition A.2.1.

**Lemma A.3.5.** *Let  $\kappa_1 > 0$  and  $B_{r_0}(x_0) \subseteq B_1$ . Suppose  $u \in \text{LSC}(Q_1)$  satisfies*

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2u) \geq 0. \quad (\text{A.3.13})$$

*There exist universal constants  $M \geq 1$  and  $\sigma > 0$  such that if  $x_0 \in A_\kappa$ ,  $\kappa \geq \kappa_1$ , then for all  $x \in B_{r_0}(x_0)$  satisfying*

$$\partial B_r(x) \cap B_{r_0}(x_0) \cap A_\kappa \neq \emptyset,$$

*we have*

$$|B_r(x) \cap A_{M\kappa} \cap B_{r_0}(x_0)| \geq \sigma |B_r(x)|.$$

*Proof.* Let  $z_1 \in \partial B_r(x) \cap B_{r_0}(x_0) \cap A_\kappa$ . By the definition of  $A_\kappa$ , we can touch  $z_1$  by a paraboloid of Hessian  $-\kappa I_d$ : there exists  $y_1 \in B_1$  such that

$$u(z_1) + \frac{\kappa}{2}|z_1 - y_1|^2 = \inf_{z \in B_1} \left( u(z) + \frac{\kappa}{2}|z - y_1|^2 \right). \quad (\text{A.3.14})$$

By some geometric considerations, we can show that there exists a ball  $B_{r_2}(x_2)$  such that

$$z_1 \in \partial B_{r_2}(x_2) \quad \text{with } x_2 \in B_r(x) \cap B_{r_0}(x_0) \text{ and } r_2 = \frac{r}{4}$$

and

$$B_{r_2/2}(x_2) \subseteq B_r(x) \cap B_{r_0}(x_0).$$

**Step 1.** We claim that there exists  $z_2 \in B_{r_2}(x_2)$  such that

$$u(z_2) + \frac{\kappa}{2}|z_2 - y_1|^2 \leq (d\Lambda + 3)\kappa\beta\theta\nu^2r^2. \quad (\text{A.3.15})$$

By applying (a properly scaled) Lemma A.3.3, there exists a barrier function  $w$  which satisfies

$$\begin{cases} \mathcal{P}_{\lambda,\Lambda}^+(D^2w) \leq -1, & \text{in } B_{r_2}(x_2) \setminus \overline{B}_{r_2/8}(x_2), \\ w = 0, & \text{on } \partial B_{r_2}(x_2), \\ 0 \leq w \leq \beta r^2, & \text{on } \partial B_{r_2/8}(x_2), \end{cases} \quad (\text{A.3.16})$$

and  $w > 0$  in  $\overline{B}_{r_2}(x_2) \setminus \overline{B}_{r_2/8}(x_2)$ . In particular, this implies that  $w(z_1) > 0$ . We have that  $w \leq \beta h$  in  $B_{r_2}(x_2) \setminus \overline{B}_{r_2/8}(x_2)$  by the maximum principle. Observe that the function

$$\varphi(z) := (d\Lambda\kappa + 2\kappa_1)w - \frac{\kappa}{2}|z - y_1|^2,$$

satisfies

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\varphi) \leq -2\kappa_1.$$

Notice that  $u$  satisfies

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2u) \geq -\kappa_1. \quad (\text{A.3.17})$$

The comparison principle implies that the map  $z \mapsto u(z) - \varphi(z)$  attains its infimum in  $B_{r_2}(x_2) \setminus \overline{B}_{r_2/8}(x_2)$  at some point  $z = z_2 \in \partial B_{r_2}(x_2) \cup \partial B_{r_2/8}(x_2)$ . It is impossible that  $z_2 \in \partial B_{r_2}(x_2)$  since (A.3.14),  $w = 0$  on  $\partial B_{r_2}(x_2)$  and  $w(z_1) > 0$  imply that

$$\begin{aligned} u(z_1) - \varphi(z_1) &= u(z_1) + \frac{\kappa}{2}|z_1 - y_1|^2 - (d\Lambda\kappa + 2\kappa_1)w(z_1) \\ &\stackrel{(\text{A.3.14})}{\leq} \inf_{z \in B_1} \left( u(z) + \frac{\kappa}{2}|z - y_1|^2 \right) \\ &\leq \inf_{z \in \partial B_r(x)} \left( u(z) + \frac{\kappa}{2}|z - y_1|^2 \right) = \inf_{z \in \partial B_r(x)} (u(z) - \varphi(z)). \end{aligned}$$

Hence  $z_2 \in \partial B_{r_2/8}(x_2)$  and so, in particular,

$$\varphi(z_2) = (d\Lambda\kappa + 2\kappa_1)w(z_2) - \frac{\kappa}{2}|z_2 - y_1|^2.$$

Using that  $w > 0$  in  $B_{r_2/2}(x_2) \setminus B_{r_2/8}(x_2)$ , we obtain that

$$\begin{aligned} u(z_1) + \frac{\kappa}{2}|z_1 - y_1|^2 &\geq u(z_1) - \varphi(z_1) \geq \inf_{z \in B_{r_2/2}(x_2) \setminus B_{r_2/8}(x_2)} (u(z) - \varphi(z)) \\ &= u(z_2) - \varphi(z_2) = u(z_2) + \frac{\kappa}{2}|z_2 - y_1|^2 - (d\Lambda\kappa + 2\kappa_1)w(z_2). \end{aligned}$$

By (A.3.16) we have  $w(z_2) \leq \beta r^2$ . Using this together with (A.3.14),  $z_1 \in \partial B_{r_2/2}(x_2)$  and  $\kappa \geq \kappa_1$ ,

$$\inf_{z \in B_1} (u(z) + \frac{\kappa}{2}|z - y_1|^2) \geq u(z_2) + \frac{\kappa}{2}|z_2 - y_1|^2 - (d\Lambda + 2)\kappa\beta r^2.$$

Recalling (A.3.14), we obtain (A.3.15).

**Step 2.** It follows that, if we set  $\gamma := 16\beta(d\Lambda + 2) + 1$ , then for every  $y_2 \in B_{r_2/8}(x_2)$ , the function

$$\psi(z) := u(z) + \frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2 \quad (\text{A.3.18})$$

satisfies  $\psi(z_2) < \min_{B_1 \setminus B_{r_2/2}(x_2)} \psi$  and therefore must attain its infimum over  $B_1$  somewhere in  $B_{r_2/2}(x_2)$ .

**Step 3.** Consider the function  $\bar{z} : B_{r_2/8}(x_2) \rightarrow B_1$  given by

$$\bar{z}(y) := \frac{1}{\gamma+1}(y_1 + \gamma y).$$

Observe by completing the square that we have for all  $z \in \mathbb{R}^d$ ,

$$\frac{\kappa}{2}|z - y_1|^2 + \frac{\gamma\kappa}{2}|z - y_2|^2 = \frac{(\gamma+1)\kappa}{2}|z - \bar{z}(y_2, s_2)|^2 + a.$$

for a certain  $a \in \mathbb{R}$ . It follows by Step 2 that the map

$$z \mapsto u(z) + \frac{(\gamma+1)\kappa}{2}|z - \bar{z}(y_2, s_2)|^2$$

attains its infimum in  $B_1$  at some point of  $B_{r_2/2}(x_2)$ . Since  $u$  satisfies (A.3.17), we can apply Lemma A.3.2 by taking  $L = \kappa_1 > 0$  and we obtain

$$|\bar{Z}(B_{r_2/8}(x_2))| \leq \frac{1}{\lambda^d} \left( \frac{\kappa_1}{(\gamma+1)\kappa} + \Lambda d \right)^d |B_{r_2/2}(x_2) \cap A_{(\gamma+1)\kappa}|. \quad (\text{A.3.19})$$

Since  $\gamma \geq 1$  and  $\gamma/(\gamma+1) \geq \frac{1}{2}$ , we deduce by the change of variables formula that

$$|\bar{Z}(B_{r_2/8}(x_2))| \geq 2^{-d} |B_{r_2/8}(x_2)|. \quad (\text{A.3.20})$$

Observe that  $B_{r_2/2}(x_2) \subseteq B_r(x_2) \cap B_{r_0}(x_0)$ . By combining this observation with (A.3.19), (A.3.20) and  $|B_{r_2/8}(x_2)| = 8^{-d}|B_{r_2}(x_2)|$ , recalling that  $\kappa \geq \kappa_1$  and taking  $M := \gamma + 1$ , we obtain that there exists a constant  $0 < \xi \leq 1$  depending only on the dimension such that

$$|B_{r_2}(x_2) \cap A_{M\kappa} \cap B_{r_0}(x_0)| \geq \xi \underbrace{\frac{2^{-4d}\lambda^d}{\left(\frac{1}{\gamma+1} + \Lambda d\right)^d}}_{:=\sigma} |B_{r_2}(x_2)|,$$

as desired.  $\square$

We next present the proof of the parabolic  $W^{2,\varepsilon}$  estimate.

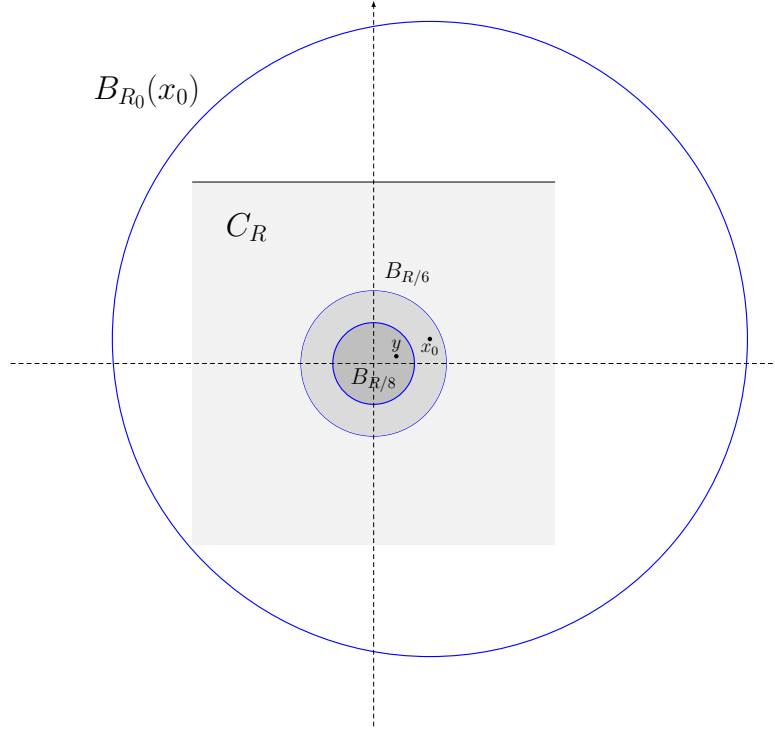


Figure A.1: The cube  $C_R$ , the balls  $B_{R/8}$ ,  $B_{R/6}$  and  $B_{R_0}(x_0)$ .

*Proof of Proposition A.2.1.* We begin with two reductions. First, we may assume that  $L = 0$ . Otherwise we replace  $u$  by  $\hat{u} := u - \frac{1}{2\lambda d}L|x|^2$ , which is solution of

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\hat{u}) \geq 0.$$

Under the assumption the estimate holds true for  $\hat{u}$ , we get

$$\begin{aligned} |\{x \in B_{1/2} : \underline{\Theta}(u, B_1)(x) > \kappa\}| &\leq |\{x \in B_{1/2} : \underline{\Theta}(\hat{u}, B_1)(x) > \kappa\}| \\ &\leq C \left( \frac{\kappa}{\sup_{B_1} |\hat{u}|} \right)^{-\varepsilon}. \end{aligned}$$

Observing that  $\sup_{B_1} |\hat{u}| \leq \sup_{B_1} |u| + \frac{1}{2\lambda d}L$ , we get the desired estimate for  $u$ .

Next, using the positive homogeneity of  $\mathcal{P}_{\lambda,\Lambda}^+$  and  $\underline{\Theta}$  and replacing  $u$  by  $\tilde{u} := u / \sup_{B_1} |u|$ , we may assume that  $\sup_{B_1} |u| = 1$ . Finally, by Lemma A.3.1, we have

$$|\{x \in B_{1/2} : \underline{\Theta}(u, B_1)(x) > \kappa\}| \leq |B_{1/2} \setminus A_\kappa|.$$

Thus, it suffices to prove that there exist some universal constants  $C, \kappa_0, \varepsilon > 0$  such that for all  $\kappa \geq \kappa_0$ ,

$$|B_{1/2} \setminus A_\kappa| \leq C \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon}. \quad (\text{A.3.21})$$

**Step 1** (Geometric configuration). We consider the cube

$$C_R(0) := (-R, R)^n,$$

with  $R > 0$  which satisfies

$$R < \frac{1}{2\sqrt{d}}, \quad (\text{A.3.22})$$

To perform our analysis, we consider two balls  $B_{R/6}(0)$  and  $B_{R/8}(0)$  (see Figure A.1) and a ball  $B_{R_0}(x_0)$  with  $x_0$  selected at Step 2 and

$$R_0 := \frac{3}{2}\sqrt{d}R. \quad (\text{A.3.23})$$

To be well-defined, our argument requires that  $B_{R/6}(0)$ ,  $B_{R/8}(0)$  and  $B_{R_0}$  are contained in  $B_1$ .

**Step 2** (Existence of the paraboloid for a certain  $\kappa = \kappa_0$ ). We claim that there exists  $x_0 \in B_{R/6}(0)$  such that there exists  $y_0 \in B_1$  such that

$$u(x_0) + \frac{\kappa}{2}|x_0 - y_0|^2 = \inf_{z \in B_1} \left( u(z) + \frac{\kappa}{2}|z - y_0|^2 \right). \quad (\text{A.3.24})$$

In addition,

$$C_R(0) \subseteq B_{R_0}(x_0) \subset B_1, \quad (\text{A.3.25})$$

and there exists  $0 < \xi < 1$ , depending only on  $\theta$ , such that

$$|C_R(0)| \geq \xi |B_{R_0}(x_0)|, \quad (\text{A.3.26})$$

To prove the claim given by (A.3.24), first we are going to find  $x_0$  realizing the infimum for a good choice of  $y_0$  and  $\kappa$  chosen sufficiently large.

First select  $y_0 \in B_{R/8}(0)$  such that

$$u(y_0) = \inf \{ u(z) : z \in B_{R/6}(0) \},$$

and  $\kappa$  such that

$$\kappa \geq \kappa_0 \quad \text{with} \quad \kappa_0 := \max \left\{ 24, \frac{2304}{R^2} \right\}. \quad (\text{A.3.27})$$

Then for all  $z \in B_1$ ,

$$u(z) + \frac{\kappa}{2}|z - y_0|^2 \geq u(y_0) - \operatorname{osc}_{B_1} u + \frac{\kappa}{2}|z - y_0|^2.$$

For all  $z \in B_1 \setminus B_{R/6}(0)$ , we have

$$\frac{1}{2}|z - y_0|^2 \geq \frac{1}{2} \left( \frac{1}{6} - \frac{1}{8} \right)^2 R^2 = \frac{1}{2} \frac{1}{576} R^2.$$

Since  $\operatorname{osc}_{Q_1} u \leq 2$ , and recalling (A.3.27), this implies that

$$\inf \left\{ u(z) + \frac{\kappa}{2}|z - y_0|^2 : z \in B_1 \setminus B_{R/6}(0) \right\} \geq u(y_0) - 2 + \frac{\kappa}{1152} R^2 \geq u(y_0).$$

Thus there exists  $x_0 \in B_{R/6}(0)$ , such that

$$u(x_0) + \frac{\kappa}{2}|x_0 - y_0|^2 = \inf_{z \in B_{R/6}} \left( u(z) + \frac{\kappa}{2}|z - y_0|^2 \right).$$

In particular, we deduce from the two last lines that

$$u(x_0) + \frac{\kappa}{2}|x_0 - y_0|^2 = \inf_{z \in B_1} \left( u(z) + \frac{\kappa}{2}|z - y_0|^2 \right).$$

To show (A.3.25), first notice that  $|x_0| \leq R/6$ . Now observe that each  $y \in C_R(0)$  satisfies

$$|y - x_0| \leq |y| + |x_0| \leq \sqrt{d}R + R/6 \leq (1 + \sqrt{d})R.$$

This means  $y \in B_{(\frac{1}{2} + \sqrt{d})R}(x_0) \subseteq B_{R_0}(x_0)$  by (A.3.23) (see also Figure A.1).

**Step 3.** By Step 2, the point  $x_0$  belongs to  $A_{\kappa_0}$ . By Lemma A.3.1, we deduce that, for all  $\kappa \geq \kappa_0$ ,  $x_0$  belongs to  $A_\kappa$ . Then we can apply Lemma A.3.4 and we get that for all  $\kappa \geq \kappa_0$ ,

$$|A_{M\kappa} \cap B_{R_0}(x_0)| \geq |B_{R_0}(x_0) \cap A_\kappa| + \sigma\eta |B_{R_0}(x_0) \setminus A_\kappa|.$$

After rearranging the terms, this implies

$$|B_{R_0}(x_0) \setminus A_{M\kappa}| \leq (1 - \sigma\eta) |B_{R_0}(x_0) \setminus A_\kappa|. \quad (\text{A.3.28})$$

**Step 4.** We claim that for all  $\kappa > \kappa_0$ , we have

$$|C_R(0) \setminus A_\kappa| \leq \frac{1}{\xi(1-\sigma\eta)} \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon} |C_R(0)|, \quad (\text{A.3.29})$$

where  $\varepsilon$  is given by  $\varepsilon := -\frac{\ln(1-\sigma\eta)}{\ln M} > 0$ . First we obtain the decay measure estimate on the ball  $B_{R_0}(x_0)$ . Let  $\kappa > \kappa_0$  and  $N$  be the integer defined by

$$M^N \kappa_0 < \kappa \leq M^{N+1} \kappa_0 \iff N := \left\lceil \frac{1}{\ln M} \ln \left( \frac{\kappa}{\kappa_0} \right) \right\rceil - 1.$$

Here  $\lceil r \rceil$  denotes, for  $r \in \mathbb{R}$ , the smallest integer not smaller than  $r$ . Then by using iteratively (A.3.28) given by Step 3, we deduce that

$$|B_{R_0}(x_0) \setminus A_\kappa| \leq (1-\sigma\eta)^N |B_{R_0}(x_0) \setminus A_{\kappa_0}|.$$

By inserting the value of  $N$ , we conclude that

$$|B_{R_0}(x_0) \setminus A_\kappa| \leq \frac{1}{1-\sigma\eta} |B_{R_0}(x_0)| \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon},$$

By combining (A.3.25) and (A.3.26) (see Step 2), we come back to the cube  $C_R(0, T_0)$  and obtain (A.3.29).

**Step 5** (Covering argument). We cover  $B_{1/2}$  by

$$B_{1/2} \subseteq \bigcup_{1 \leq i \leq N} \overline{C}_R(X_i) \subset Q_1$$

where the cylinders  $C_R(X_i) := C_R(0) + X_i$  are disjoint. Then

$$|B_{1/2} \setminus A_\kappa| \leq \sum_{i=1}^N |C_R(X_i) \setminus A_\kappa|.$$

By applying Step 4 to each cylinder  $C_R(X_i)$ , we get

$$|B_{1/2} \setminus A_\kappa| \leq \frac{1}{\xi(1-\sigma\eta)} \sum_{i=1}^N |C_R(X_i)| \left( \frac{\kappa}{\kappa_0} \right)^{-\varepsilon}.$$

Since  $\sum_{i=1}^N |C_R(X_i)| \leq |B_1|$ , we get the estimate given by (A.3.21). □

# Annexe B

## Poster

Poster présenté lors de la session de posters à la conférence :

*Journées Lions-Magenes,  
Laboratoire Jacques-Louis Lions,  
Université Pierre et Marie Curie, Paris*

Ce poster présente l'interprétation par jeu, détaillé largement au chapitre 1 dans le cas général, dans le cas parabolique où la non-linéarité  $\mathcal{F}$  ne présente pas de dépendance par la variable  $u$ . L'EDP admet alors toujours dans ce cas un principe de comparaison et la fonction score  $u^\varepsilon$  satisfait une équation de programmation dynamique. L'EDP étudiée ici a la forme suivante :

$$\begin{cases} \partial_t u - \mathcal{F}(t, x, Du, D^2 u) = 0, & \text{pour } x \in \Omega \text{ et } t < T, \\ \langle Du(x, t), n(x) \rangle = h(x), & \text{pour } x \in \Omega \text{ et } t < T, \\ u(x, T) = g(x), & \text{pour } x \in \overline{\Omega}. \end{cases}$$



# A game interpretation for fully nonlinear parabolic and elliptic equations with a Neumann boundary condition

Jean-Paul Daniel

UPMC Univ 06, UMR 7598, Laboratoire J.L. Lions, F-75005 Paris, France

## Motivation

- Deterministic control interpretation via "two persons repeated games" for a broad class of fully nonlinear equations: (1)
  - Parabolic setting : the whole space case.
  - Elliptic setting : the Dirichlet problem.
- Motivation : to adapt their approach to the Neumann problem in both settings. In the parabolic case we can study

$$\begin{cases} \partial_t u - f(t, x, Du, D^2u) = 0, & \text{for } x \in \Omega \text{ and } t < T, \\ \langle Du(x, t), n(x) \rangle = h(x) & \text{for } x \in \partial\Omega \text{ and } t < T, \\ u(x, T) = g(x) & \text{for } x \in \bar{\Omega}, \end{cases} \quad (1)$$

where  $\Omega$  is a  $C^2$ -boundary domain and  $f$  is elliptic:

$$f(t, x, p, \Gamma_1 + \Gamma_2) \leq f(t, x, p, \Gamma_1) \text{ when } \Gamma_2 \geq 0.$$

- The oblique and mixed type Dirichlet-Neumann boundary conditions can also be treated by this analysis.

## The game

- Two players Mark and Helen in a repeated game.
- Let  $\alpha, \beta, \gamma \in [0, 1]$  satisfy some algebraic relations.

### Rules of the game

Helen's goal is to maximize her score at time  $T$ , and Mark's is to work against her. If at time  $t_j = t_0 + j\varepsilon^2$ , the position is  $x_j$  and Helen's score is  $y_j$ , then

1. Helen chooses a vector  $p_j \in \mathbb{R}^N$  and an  $N \times N$  symmetric matrix  $\Gamma_j$ , restricted by

$$\|p_j\| \leq \varepsilon^{-\beta}, \|\Gamma_j\| \leq \varepsilon^{-\gamma}.$$

2. Taking Helen's choice into account, Mark chooses the stock price  $x_{j+1}$  so as to degrade Helen's outcome. Mark chooses an intermediate point  $\tilde{x}_{j+1} \in \mathbb{R}^N$  such that

$$\|\Delta \tilde{x}_j\| \leq \varepsilon^{1-\alpha}.$$

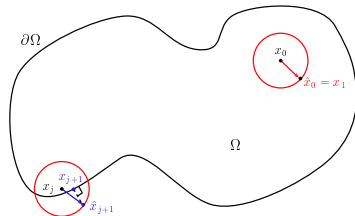
This position  $\tilde{x}_{j+1}$  determines the new position  $x_{j+1} = x_j + \Delta x_j \in \bar{\Omega}$  at time  $t_{j+1}$  by the rule

$$x_{j+1} = \text{proj}_{\bar{\Omega}}(\tilde{x}_{j+1}).$$

3. Helen's score changes to

$$y_{j+1} = y_j - p_j \cdot \Delta \tilde{x}_j - \frac{1}{2} \langle \Gamma_j \Delta \tilde{x}_j, \Delta \tilde{x}_j \rangle - \varepsilon^2 f(t_j, x_j, p_j, \Gamma_j) + \|\Delta \tilde{x}_j - \Delta x_j\| h(x_j + \Delta x_j).$$

4. The clock steps forward to  $t_{j+1} = t_j + \varepsilon^2$ : the process repeats, stopping when  $t_K = T$ . At  $t_K = T$ , Helen receives  $g(x_K)$ .



### Dynamic programming principle

Helen's value function  $u^\varepsilon$  is determined by:

- the final-time condition  $u^\varepsilon(x, T) = g(x)$ ,
- and the dynamic programming principle:

$$u^\varepsilon(x, t_j) = \max_{p \in \mathbb{R}^N} \min_{\tilde{x}} [u^\varepsilon(x + \Delta x, t_{j+1}) - p_j \cdot \Delta \tilde{x}_j - \frac{1}{2} \langle \Gamma_j \Delta \tilde{x}_j, \Delta \tilde{x}_j \rangle - \varepsilon^2 f(t_j, x_j, p_j, \Gamma_j) + \|\Delta \tilde{x}_j - \Delta x_j\| h(x_j + \Delta x_j)] \quad (2)$$

with the constraints  $\|p\| \leq \varepsilon^{-\beta}$ ,  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ , and  $\|\Delta \tilde{x}\| \leq \varepsilon^{1-\alpha}$ .

## Main result

**Theorem 1.** Consider the final-value problem (1). If the PDE has a **comparison principle** (for uniformly bounded solutions) then it follows that  $u^\varepsilon$  converge locally uniformly to the unique viscosity solution of (1).

### Viscosity solutions for the parabolic PDE

**Definition 2.** A real-valued lower-semicontinuous function  $u(x, t)$  defined on  $\bar{\Omega} \times [0, T]$  is a **viscosity supersolution** (resp. **subsolution**) of the final value problem (1) if

1. for any  $(x_0, t_0)$  with  $x_0 \in \Omega$  and  $t_0 \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local **minimum** at  $x_0$ , we have

$$\partial_t \phi(x_0, t_0) - f(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)) \geq 0,$$

2. for any  $(x_0, t_0)$  with  $x_0 \in \partial\Omega$  and  $t_0 \leq t_0 < T$  and any smooth  $\phi(x, t)$  such that  $u - \phi$  has a local **minimum** on  $\bar{\Omega}$  at  $x_0$ , we have

$$\min \left\{ -(\partial_t \phi(x_0, t_0) - f(t_0, x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0))), \right.$$

$$\left. \langle D\phi(x_0, t_0), n(x_0) \rangle - h(x_0) \right\} \geq 0,$$

3.  $u \leq g$  at the final time  $t = T$ .

A function  $u$  on  $\bar{\Omega} \times [0, T]$  is a **viscosity solution** of (1) if it is both a viscosity subsolution and a viscosity supersolution of (1).

### Comparison principle

When we construct some good subsolution  $u$  and supersolution  $v$  of the PDE, can we compare  $u$  and  $v$ ?

- $u \geq v$ : satisfied by construction.
- Reverse inequality  $\rightarrow$  **comparison principle**:

If  $u$  is a subsolution and  $v$  is a supersolution then  $u \leq v$ .

If the PDE has a comparison principle:

$\Rightarrow u = v$  is a viscosity solution of the PDE.

$\Rightarrow$  unique since the comparison principle implies uniqueness.

By the dynamic programming principle (2),  $u^\varepsilon$  satisfy

$$u^\varepsilon(x, t) = S_\varepsilon \left[ x, t, u^\varepsilon(\cdot, t + \varepsilon^2) \right],$$

where  $S_\varepsilon[x, t, \phi]$  is defined for any  $x \in \bar{\Omega}$  and  $t \leq T$  and any continuous function  $\phi: \bar{\Omega} \rightarrow \mathbb{R}$  by

$$S_\varepsilon[x, t, \phi] = \max_{p \in \mathbb{R}^N} \min_{\tilde{x}} \left[ \phi(x + \Delta x) - \left( p \cdot \Delta \tilde{x} + \frac{1}{2} \langle \Gamma \Delta \tilde{x}, \Delta \tilde{x} \rangle + \varepsilon^2 f(t, x, p, \Gamma) - \|\Delta \tilde{x} - \Delta x\| h(x + \Delta x) \right) \right]$$

subject to the usual constraints  $\|p\| \leq \varepsilon^{-\beta}$ ,  $\|\Gamma\| \leq \varepsilon^{-\gamma}$ , and  $\|\Delta \tilde{x}\| \leq \varepsilon^{1-\alpha}$ .

## Sketch of the rigorous proof

Landmark theorem of Barles and Souganidis [2] states:

"If a numerical scheme is monotone, stable, and consistent for solving our PDE if for any  $C^\infty$  function  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$  and any  $x \in \mathbb{R}^n$  and  $t \leq T$  we have

### Monotonicity

$S_\varepsilon$  is monotone, i.e. if  $\phi_1 \leq \phi_2$ , then

$$S_\varepsilon[x, t, \phi_1] \leq S_\varepsilon[x, t, \phi_2].$$

### Consistency

The time-stepping scheme  $\phi(x, t) = S_\varepsilon[x, t, \phi(\cdot, t + \varepsilon^2)]$  is consistent for solving our PDE if for any  $C^\infty$  function  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$  and any  $x \in \mathbb{R}^n$  and  $t \leq T$  we have

$$\lim_{\varepsilon \rightarrow 0} \frac{S_\varepsilon[x, t, \phi(\cdot, t + \varepsilon^2)] - \phi(x, t)}{\varepsilon^2} \text{ exists}$$

and corresponds to the conditions given by Definition 2.

- $x \in \Omega$ :  $\lim_{\varepsilon \rightarrow 0} \frac{S_\varepsilon[x, t, \phi(\cdot, t + \varepsilon^2)] - \phi(x, t)}{\varepsilon^2} = -f(t, x, D\phi(x, t), D^2\phi(x, t))$ .
- $x \in \partial\Omega$ : degeneration  $\Rightarrow$  find lower and upper bounds of  $S_\varepsilon[x, t, \phi] - \phi(x, t)$  depending on
  - the geometry of the boundary  $\partial\Omega$ ,
  - the distance to the boundary  $d(x) = d(x, \partial\Omega)$ ,
  - the size of  $h - D\phi \cdot n$  on a neighborhood of  $x$ .

## Formal derivation of the PDE

Heuristic hypothesis:

- $\partial\Omega \approx$  hyperplan tangent at  $\tilde{x} = \text{proj}_{\partial\Omega}(x)$ .
- We suppress the dependence of  $u^\varepsilon$  on  $\varepsilon$ .

We formally obtain

$$u(x, t) \approx S_\varepsilon \left[ x, t, u(\cdot, t + \varepsilon^2) \right].$$

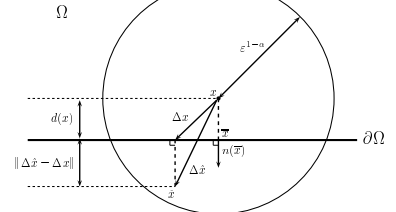
If  $u$  is smooth we obtain by Taylor expansion that

$$\begin{aligned} u(x + \Delta x, t + \varepsilon^2) &= \Delta x \cdot \nabla u(x, t) + \frac{\varepsilon^2}{2} \langle D^2 u(x, t), \Delta x, \Delta x \rangle \\ &= u(x, t) + \varepsilon^2 u_t + Du \cdot \Delta x + \frac{1}{2} \langle D^2 u(x, t), \Delta x, \Delta x \rangle \\ &\quad + \|\Delta x - \Delta x\| \underbrace{h(x + \Delta x) - Du \cdot n(x + \Delta x)}_{\approx m(\text{loc. constant})}. \end{aligned}$$

So it is sufficient to examine :

$$0 \approx \varepsilon^2 u_t + \max_{p \in \mathbb{R}^N} \min_{\tilde{x}} \left[ \underbrace{(Du - p) \cdot \Delta x + \|\Delta x - \Delta x\| m}_{\text{first order in } \Delta x} - \frac{1}{2} \langle \Gamma \Delta x, \Delta x \rangle - \varepsilon^2 f(t, x, p, \Gamma) \right].$$

second order in  $\Delta x$



To determine the optimal choice for Helen on  $p$ , we consider the order- $\varepsilon$  optimization problem obtained by neglecting the second order- $\varepsilon$  terms

$$\mathcal{M} = \max_p \min_{\tilde{x}} \left[ (Du - p) \cdot \Delta x + \|\Delta x - \Delta x\| m \right].$$

If  $m > 0$ , Helen chooses  $p = Du$  whereas if  $m \leq 0$ , Helen chooses

$$p = Du + \frac{m}{2} \left( 1 - \frac{d(x)}{\varepsilon^{1-\alpha}} \right) n(\tilde{x}).$$

A further analysis provides

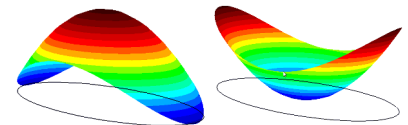
$$0 \approx \varepsilon^2 u_t + \begin{cases} \frac{1}{2} (\varepsilon^{1-\alpha} - d(x)) m - \varepsilon^2 f(t, x, p_{\text{opt}}, \Gamma_{\text{opt}}) + o(\varepsilon^2) & \text{if } d(x) \leq \varepsilon^{1-\alpha} \text{ and } m \leq 0, \\ -\varepsilon^2 f(t, x, Du, D^2u) & \text{if } d(x) \geq \varepsilon^{1-\alpha} \text{ or } m > 0. \end{cases}$$

We distinguish two cases:

- If  $x \in \Omega$ , we get  $u_t - f(t, x, Du, D^2u) = 0$ .
- If  $x \in \bar{\Omega}$  ( $d(x) = 0$ ), by dividing by  $\varepsilon^{1-\alpha}$  and letting  $\varepsilon \downarrow 0$ , we get  $m = 0$ .

### Stability

- Show that if the final-time data are uniformly bounded, then  $u^\varepsilon(x, t)$  remains bounded when  $\varepsilon \downarrow 0$  (no blow-up).
- In our case: construct good subsolutions and supersolutions which take care of the Neumann condition at the boundary.



Heat equation in a disk :  $h = -1$  on the left,  $h = 1$  on the right.

## References

- [1] Sylvia Serfaty and Robert V. Kohn. A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. 2010.
- [2] G. Barles and P. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. 1991.
- [3] Y. Giga and Q. Liu. A billiard-based game interpretation of the Neumann problem for the curve shortening equation. 2009.

# Bibliographie

- [1] Luis Alvarez, Frédéric Guichard, Pierre-Louis Lions, and Jean-Michel Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123(3) :199–257, 1993.
- [2] Scott N. Armstrong, Luis E. Silvestre, and Charles K. Smart. Partial regularity of solutions of fully nonlinear, uniformly elliptic equations. *Comm. Pure Appl. Math.*, 65(8) :1169–1184, 2012.
- [3] Scott N. Armstrong and Charles K. Smart. A finite difference approach to the infinity laplace equation and tug-of-war games. *Trans. Amer. Math. Soc.*, 364(2) :595–636, 2012.
- [4] Scott N. Armstrong and Charles K. Smart. Quantitative stochastic homogenization of elliptic equations in nondivergence form. *Archive for Rational Mechanics and Analysis*, 214(3) :867–911, 2014.
- [5] Scott N. Armstrong and Charles K. Smart. Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity. *The Annals of Probability*, 42(6) :2558–2594, 11 2014.
- [6] Scott N. Armstrong, Charles K. Smart, and Stephanie J. Somersille. An infinity Laplace equation with gradient term and mixed boundary conditions. *Proc. Amer. Math. Soc.*, 139(5) :1763–1776, 2011.
- [7] Scott N. Armstrong and Maxim Trokhimtchouk. Long-time asymptotics for fully nonlinear homogeneous parabolic equations. *Calc. Var. Partial Differential Equations*, 38(3-4) :521–540, 2010.
- [8] Martino Bardi and Italo Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control : Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.
- [9] G. Barles, L. Bronsard, and P. E. Souganidis. Front propagation for reaction-diffusion equations of bistable type. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(5) :479–496, 1992.
- [10] Guy Barles. Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations. *J. Differential Equations*, 106(1) :90–106, 1993.
- [11] Guy Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1994.
- [12] Guy Barles. Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications. *J. Differential Equations*, 154(1) :191–224, 1999.
- [13] Guy Barles and Jérôme Busca. Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. *Comm. Partial Differential Equations*, 26(11-12) :2323–2337, 2001.
- [14] Guy Barles and Christine Georgelin. A simple proof of convergence for an approximation scheme for computing motions by mean curvature. *SIAM J. Numer. Anal.*, 32(2) :484–500, 1995.
- [15] Guy Barles and Espen R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1) :33–54, 2002.
- [16] Guy Barles and Espen R. Jakobsen. Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations. *SIAM J. Numer. Anal.*, 43(2) :540–558 (electronic), 2005.

- [17] Guy Barles and Pierre-Louis Lions. Remarques sur les problèmes de réflexion oblique. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(1) :69–74, 1995.
- [18] Guy Barles and B. Perthame. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5) :1133–1148, 1988.
- [19] Guy Barles and E. Rouy. A strong comparison result for the Bellman equation arising in stochastic exit time control problems and its applications. *Comm. Partial Differential Equations*, 23(11-12) :1995–2033, 1998.
- [20] Guy Barles and Panagiotis E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3) :271–283, 1991.
- [21] Guy Barles and Panagiotis E. Souganidis. A new approach to front propagation problems : theory and applications. *Arch. Rational Mech. Anal.*, 141(3) :237–296, 1998.
- [22] Samuel Biton, Pierre Cardaliaguet, and Olivier Ley. Nonfattening condition for the generalized evolution by mean curvature and applications. *Interfaces Free Bound.*, 10(1) :1–14, 2008.
- [23] Clayton Bjorland, Luis Caffarelli, and Alessio Figalli. Non-local gradient dependent operators. *Adv. Math.*, 230(4-6) :1859–1894, 2012.
- [24] Clayton Bjorland, Luis Caffarelli, and Alessio Figalli. Nonlocal tug-of-war and the infinity fractional Laplacian. *Comm. Pure Appl. Math.*, 65(3) :337–380, 2012.
- [25] J. Frédéric Bonnans, Stefania Maroso, and Housnaa Zidani. Error estimates for stochastic differential games : the adverse stopping case. *IMA J. Numer. Anal.*, 26(1) :188–212, 2006.
- [26] Rainer Buckdahn, Pierre Cardaliaguet, and Marc Quincampoix. A representation formula for the mean curvature motion. *SIAM J. Math. Anal.*, 33(4) :827–846 (electronic), 2001.
- [27] Luis Caffarelli. Elliptic second order equations. *Rend. Sem. Mat. Fis. Milano*, 58 :253–284 (1990), 1988.
- [28] Luis Caffarelli, Michael G. Crandall, M. Kocan, and A. Święch. On viscosity solutions of fully nonlinear equations with measurable ingredients. *Comm. Pure Appl. Math.*, 49(4) :365–397, 1996.
- [29] Luis Caffarelli, Joseph J. Kohn, Louis Nirenberg, and Joel Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations. *Comm. Pure Appl. Math.*, 38(2) :209–252, 1985.
- [30] Luis Caffarelli, Louis Nirenberg, and Joel Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3) :369–402, 1984.
- [31] Luis A. Caffarelli. Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math. (2)*, 130(1) :189–213, 1989.
- [32] Luis A. Caffarelli. Interior  $W^{2,p}$  estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)*, 131(1) :135–150, 1990.
- [33] Luis A. Caffarelli. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math. (2)*, 131(1) :129–134, 1990.
- [34] Luis A. Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.
- [35] Luis A. Caffarelli and Panagiotis E. Souganidis. A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs. *Comm. Pure Appl. Math.*, 61(1) :1–17, 2008.
- [36] Luis A. Caffarelli and Panagiotis E. Souganidis. Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media. *Invent. Math.*, 180(2) :301–360, 2010.
- [37] Luis A. Caffarelli, Panagiotis E. Souganidis, and Lihe Wang. Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Comm. Pure Appl. Math.*, 58(3) :319–361, 2005.
- [38] Pierre Cardaliaguet. Long time average of first order mean field games and weak KAM theory. *Dyn. Games Appl.*, 3(4) :473–488, 2013.

- 
- [39] Pierre Cardaliaguet. Weak solutions for first order mean field games with local coupling. May 2013.
  - [40] Elisabetta Carlini, Maurizio Falcone, and Roberto Ferretti. Convergence of a large time-step scheme for mean curvature motion. *Interfaces Free Bound.*, 12(4) :409–441, 2010.
  - [41] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3) :749–786, 1991.
  - [42] Patrick Cheridito, H. Mete Soner, Nizar Touzi, and Nicolas Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm. Pure Appl. Math.*, 60(7) :1081–1110, 2007.
  - [43] M. G. Crandall and P.-L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Math. Comp.*, 43(167) :1–19, 1984.
  - [44] Michael G. Crandall, Lawrence C. Evans, and Pierre-Louis Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 282(2) :487–502, 1984.
  - [45] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1) :1–67, 1992.
  - [46] Michael G. Crandall, M. Kocan, and A. Świąch.  $L^p$ -theory for fully nonlinear uniformly parabolic equations. *Comm. Partial Differential Equations*, 25(11-12) :1997–2053, 2000.
  - [47] Michael G. Crandall and Pierre-Louis Lions. Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(3) :183–186, 1981.
  - [48] Michael G. Crandall and Pierre-Louis Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1) :1–42, 1983.
  - [49] Michael G. Crandall and Pierre-Louis Lions. Convergent difference schemes for nonlinear parabolic equations and mean curvature motion. *Numer. Math.*, 75(1) :17–41, 1996.
  - [50] Jean-Paul Daniel. A game interpretation of the Neumann problem for fully nonlinear parabolic and elliptic equations. *ESAIM Control Optim. Calc. Var.*, 19(4) :1109–1165, 2013.
  - [51] Jean-Paul Daniel. Quadratic expansions and partial regularity for fully nonlinear uniformly parabolic equations. *Calculus of Variations and Partial Differential Equations*, pages 1–34, 2014.
  - [52] Guido De Philippis and Alessio Figalli. Second order stability for the Monge–Ampère equation and strong Sobolev convergence of optimal transport maps. *Anal. PDE*, 6(4) :993–1000, 2013.
  - [53] Guido De Philippis and Alessio Figalli.  $W^{2,1}$  regularity for solutions of the Monge–Ampère equation. *Invent. Math.*, 192(1) :55–69, 2013.
  - [54] Guido De Philippis and Alessio Figalli. The Monge–Ampère equation and its link to optimal transportation. *Bull. Amer. Math. Soc. (N.S.)*, 51(4) :527–580, 2014.
  - [55] Guido De Philippis, Alessio Figalli, and Ovidiu Savin. A note on interior  $W^{2,1+\varepsilon}$  estimates for the Monge–Ampère equation. *Math. Ann.*, 357(1) :11–22, 2013.
  - [56] Hongjie Dong and Nicolai V. Krylov. The rate of convergence of finite-difference approximations for parabolic Bellman equations with Lipschitz coefficients in cylindrical domains. *Appl. Math. Optim.*, 56(1) :37–66, 2007.
  - [57] Paul Dupuis and Hitoshi Ishii. SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.*, 21(1) :554–580, 1993.
  - [58] Lawrence C. Evans. A convergence theorem for solutions of nonlinear second-order elliptic equations. *Indiana Univ. Math. J.*, 27(5) :875–887, 1978.
  - [59] Lawrence C. Evans. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel J. Math.*, 36(3-4) :225–247, 1980.
  - [60] Lawrence C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, 35(3) :333–363, 1982.
  - [61] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

- [62] Lawrence C. Evans and P. E. Souganidis. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana Univ. Math. J.*, 33(5) :773–797, 1984.
- [63] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3) :635–681, 1991.
- [64] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.*, 330(1) :321–332, 1992.
- [65] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. III. *J. Geom. Anal.*, 2(2) :121–150, 1992.
- [66] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. IV. *J. Geom. Anal.*, 5(1) :77–114, 1995.
- [67] W. H. Fleming and M. Nisio. Differential games for stochastic partial differential equations. *Nagoya Math. J.*, 131 :75–107, 1993.
- [68] Wendell H. Fleming and H. Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.
- [69] Avner Friedman. Differential games. In *Handbook of game theory with economic applications*, Vol. II, volume 11 of *Handbooks in Econom.*, pages 781–799. North-Holland, Amsterdam, 1994.
- [70] Brittany D. Froese and Adam M. Oberman. Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampère equation in dimensions two and higher. *SIAM J. Numer. Anal.*, 49(4) :1692–1714, 2011.
- [71] Brittany D. Froese and Adam M. Oberman. Convergent filtered schemes for the Monge-Ampère partial differential equation. *SIAM J. Numer. Anal.*, 51(1) :423–444, 2013.
- [72] Yoshikazu Giga. *Surface evolution equations*, volume 99 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006. A level set approach.
- [73] Yoshikazu Giga and Qing Liu. A billiard-based game interpretation of the Neumann problem for the curve shortening equation. *Adv. Differential Equations*, 14(3-4) :201–240, 2009.
- [74] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [75] Cristian E. Gutiérrez. *The Monge-Ampère equation*. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA, 2001.
- [76] Cyril Imbert. Level set approach for fractional mean curvature flows. *Interfaces Free Bound.*, 11(1) :153–176, 2009.
- [77] Cyril Imbert and Sylvia Serfaty. Repeated games for non-linear parabolic integro-differential equations and integral curvature flows. *Discrete Contin. Dyn. Syst.*, 29(4) :1517–1552, 2011.
- [78] Cyril Imbert and Luis Silvestre. Introduction to fully nonlinear parabolic equations. [cyrilimbert.files.wordpress.com/2012/06/pln.pdf](http://cyrilimbert.files.wordpress.com/2012/06/pln.pdf), June 2012.
- [79] Hitoshi Ishii. Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDEs. *Duke Math. J.*, 62(3) :633–661, 1991.
- [80] N. Ivochkina and O. Ladyzhenskaya. On parabolic problems generated by some symmetric functions of the eigenvalues of the Hessian. *Topol. Methods Nonlinear Anal.*, 4(1) :19–29, 1994.
- [81] N. Ivochkina and O. Ladyzhenskaya. Flows generated by symmetric functions of the eigenvalues of the Hessian. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 221(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 26) :127–144, 258, 1995.
- [82] N. M. Ivochkina and O. A. Ladyzhenskaya. The first initial-boundary value problem for evolution equations generated by traces of order  $m$  of the Hessian of the unknown surface. *Dokl. Akad. Nauk*, 337(3) :300–303, 1994.
- [83] N. M. Ivochkina and O. A. Ladyzhenskaya. Parabolic equations generated by symmetric functions of the eigenvalues of the Hessian or by the principal curvatures of a surface. I. Parabolic Monge-Ampère equations. *Algebra i Analiz*, 6(3) :141–160, 1994.

- 
- [84] Espen R. Jakobsen. On error bounds for approximation schemes for non-convex degenerate elliptic equations. *BIT*, 44(2) :269–285, 2004.
  - [85] Espen R. Jakobsen. On error bounds for monotone approximation schemes for multi-dimensional Isaacs equations. *Asymptot. Anal.*, 49(3-4) :249–273, 2006.
  - [86] Robert Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Archive for Rational Mechanics and Analysis*, 101 :1–27, 1988.
  - [87] Robert V. Kohn and Sylvia Serfaty. A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.*, 59(3) :344–407, 2006.
  - [88] Robert V. Kohn and Sylvia Serfaty. A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. *Comm. Pure Appl. Math.*, 63(10) :1298–1350, 2010.
  - [89] Nicolai V. Krylov. *Controlled diffusion processes*, volume 14 of *Applications of Mathematics*. Springer-Verlag, New York, 1980. Translated from the Russian by A. B. Aries.
  - [90] Nicolai V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(3) :487–523, 670, 1982.
  - [91] Nicolai V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(1) :75–108, 1983.
  - [92] Nicolai V. Krylov. *Nonlinear elliptic and parabolic equations of the second order*, volume 7 of *Mathematics and its Applications (Soviet Series)*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzitsky [P. L. Buzitskiĭ].
  - [93] Nicolai V. Krylov. On the rate of convergence of finite-difference approximations for Bellman’s equations. *Algebra i Analiz*, 9(3) :245–256, 1997.
  - [94] Nicolai V. Krylov. On the rate of convergence of finite-difference approximations for Bellman’s equations with variable coefficients. *Probab. Theory Related Fields*, 117(1) :1–16, 2000.
  - [95] Nicolai V. Krylov. The rate of convergence of finite-difference approximations for Bellman equations with Lipschitz coefficients. *Appl. Math. Optim.*, 52(3) :365–399, 2005.
  - [96] Nicolai V. Krylov and Mikhail V. Safonov. An estimate for the probability of a diffusion process hitting a set of positive measure. *Dokl. Akad. Nauk SSSR*, 245(1) :18–20, 1979.
  - [97] Nicolai V. Krylov and Mikhail V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1) :161–175, 239, 1980.
  - [98] Hung Ju Kuo and Neil S. Trudinger. Discrete methods for fully nonlinear elliptic equations. *SIAM J. Numer. Anal.*, 29(1) :123–135, 1992.
  - [99] Hung-Ju Kuo and Neil S. Trudinger. Schauder estimates for fully nonlinear elliptic difference operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(6) :1395–1406, 2002.
  - [100] Jean-Michel Lasry and Pierre-Louis Lions. A remark on regularization in Hilbert spaces. *Israel J. Math.*, 55(3) :257–266, 1986.
  - [101] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9) :619–625, 2006.
  - [102] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10) :679–684, 2006.
  - [103] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Jpn. J. Math.*, 2(1) :229–260, 2007.
  - [104] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
  - [105] Fang-Hua Lin. Second derivative  $L^p$ -estimates for elliptic equations of nondivergent type. *Proc. Amer. Math. Soc.*, 96(3) :447–451, 1986.
  - [106] Pierre-Louis Lions. Neumann type boundary conditions for Hamilton-Jacobi equations. *Duke Math. J.*, 52(4) :793–820, 1985.
  - [107] Pierre-Louis Lions, José-Luis Menaldi, and Alain-Sol Sznitman. Construction de processus de diffusion réfléchis par pénalisation du domaine. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(11) :559–562, 1981.

- [108] Pierre-Louis Lions and Alain-Sol Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37(4) :511–537, 1984.
- [109] Qing Liu. On game interpretations for the curvature flow equation and its boundary problems. *RIMS Kokyuroku, University of Kyoto*, 1633 :138–150, 2009.
- [110] Qing Liu. Fattening and comparison principle for level-set equations of mean curvature type. *SIAM J. Control Optim.*, 49(6) :2518–2541, 2011.
- [111] Jun Luo and Nicolai V. Krylov. On the rate of convergence of the finite-difference approximations for parabolic Bellman equations with constant coefficients. *Appl. Math. Optim.*, 58(3) :315–344, 2008.
- [112] Gaspard Monge. Mémoire sur la théorie des déblais et des remblais. *Hist. de l'Acad. royale des sciences*, pages 666–705, 1781.
- [113] Nikolai Nadirashvili and Serge Vlăduț. Nonclassical solutions of fully nonlinear elliptic equations. *Geom. Funct. Anal.*, 17(4) :1283–1296, 2007.
- [114] Nikolai Nadirashvili and Serge Vlăduț. Singular solutions of Hessian fully nonlinear elliptic equations. *Adv. Math.*, 228(3) :1718–1741, 2011.
- [115] Nikolai Nadirashvili and Serge Vlăduț. Homogeneous solutions of fully nonlinear elliptic equations in four dimensions. *Comm. Pure Appl. Math.*, 66(10) :1653–1662, 2013.
- [116] Nikolai Nadirashvili and Serge Vlăduț. Singular solutions of Hessian elliptic equations in five dimensions. *J. Math. Pures Appl. (9)*, 100(6) :769–784, 2013.
- [117] Makiko Nisio. Stochastic differential games and viscosity solutions of Isaacs equations. *Nagoya Math. J.*, 110 :163–184, 1988.
- [118] Makiko Nisio. Optimal control for stochastic partial differential equations and viscosity solutions of Bellman equations. *Nagoya Math. J.*, 123 :13–37, 1991.
- [119] Adam M. Oberman. A convergent monotone difference scheme for motion of level sets by mean curvature. *Numer. Math.*, 99(2) :365–379, 2004.
- [120] Adam M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations : Hamilton-Jacobi equations and free boundary problems. *SIAM J. Numer. Anal.*, 44(2) :879–895 (electronic), 2006.
- [121] Yuval Peres, Oded Schramm, Scott Sheffield, and David B. Wilson. Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.*, 22(1) :167–210, 2009.
- [122] Huy  n Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2009.
- [123] Mikhail V. Safonov. The classical solution of the elliptic Bellman equation. *Dokl. Akad. Nauk SSSR*, 278(4) :810–813, 1984.
- [124] Mikhail V. Safonov. Classical solution of second-order nonlinear elliptic equations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(6) :1272–1287, 1328, 1988.
- [125] Moto-Hiko Sato. Interface evolution with Neumann boundary condition. *Adv. Math. Sci. Appl.*, 4(1) :249–264, 1994.
- [126] Ovidiu Savin. Small perturbation solutions for elliptic equations. *Comm. Partial Differential Equations*, 32(4-6) :557–578, 2007.
- [127] Ovidiu Savin. Regularity of flat level sets in phase transitions. *Ann. of Math. (2)*, 169(1) :41–78, 2009.
- [128] Natasa Sesum. Rate of convergence of the mean curvature flow. *Comm. Pure Appl. Math.*, 61(4) :464–485, 2008.
- [129] H. Mete Soner. Motion of a set by the curvature of its boundary. *J. Differential Equations*, 101(2) :313–372, 1993.
- [130] H. Mete Soner and Nizar Touzi. A stochastic representation for mean curvature type geometric flows. *Ann. Probab.*, 31(3) :1145–1165, 2003.

- 
- [131] Pierpaolo Soravia. Generalized motion of a front propagating along its normal direction : a differential games approach. *Nonlinear Anal.*, 22(10) :1247–1262, 1994.
  - [132] Pierpaolo Soravia. Estimates of convergence of fully discrete schemes for the Isaacs equation of pursuit-evasion differential games via maximum principle. *SIAM J. Control Optim.*, 36(1) :1–11 (electronic), 1998.
  - [133] Panagiotis E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 59(1) :1–43, 1985.
  - [134] Panagiotis E. Souganidis. Interface dynamics in phase transitions. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1133–1144. Birkhäuser, Basel, 1995.
  - [135] Panagiotis E. Souganidis. Front propagation : theory and applications. In *Viscosity solutions and applications (Montecatini Terme, 1995)*, volume 1660 of *Lecture Notes in Math.*, pages 186–242. Springer, Berlin, 1997.
  - [136] Joel Spencer. Balancing games. *J. Combinatorial Theory Ser. B*, 23(1) :68–74, 1977.
  - [137] Caroline Sweezy. The Hausdorff dimension of elliptic and elliptic-caloric measure in  $\mathbf{R}^N$ ,  $N \geq 3$ . *Proc. Amer. Math. Soc.*, 121(3) :787–793, 1994.
  - [138] Hiroshi Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, 9(1) :163–177, 1979.
  - [139] Daniel Tataru. Viscosity solutions for the dynamic programming equations. *Appl. Math. Optim.*, 25(2) :109–126, 1992.
  - [140] Neil S. Trudinger. Hölder gradient estimates for fully nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 108(1-2) :57–65, 1988.
  - [141] Neil S. Trudinger. On regularity and existence of viscosity solutions of nonlinear second order, elliptic equations. In *Partial differential equations and the calculus of variations, Vol. II*, volume 2 of *Progr. Nonlinear Differential Equations Appl.*, pages 939–957. Birkhäuser Boston, Boston, MA, 1989.
  - [142] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.
  - [143] Lihe Wang. On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.*, 45(1) :27–76, 1992.
  - [144] Lihe Wang. On the regularity theory of fully nonlinear parabolic equations. II. *Comm. Pure Appl. Math.*, 45(2) :141–178, 1992.
  - [145] Lihe Wang. On the regularity theory of fully nonlinear parabolic equations. III. *Comm. Pure Appl. Math.*, 45(3) :255–262, 1992.
  - [146] Yu Wang. Small perturbation solutions for parabolic equations. *Indiana Univ. Math. J.*, 62(2) :671–697, 2013.
  - [147] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1999. Hamiltonian systems and HJB equations.





## Résumé

Dans cette thèse, nous nous intéressons à des résultats d'approximation et de régularité pour des solutions de viscosité d'équations elliptiques et paraboliques non-linéaires.

Dans le premier chapitre, nous considérons une classe générale d'équations elliptiques et paraboliques non-linéaires munies de conditions de Neumann inhomogènes. Nous proposons une interprétation de contrôle déterministe par des jeux répétés à deux personnes qui consiste à représenter la solution comme la limite de la suite des scores associés aux jeux. Le premier joueur souhaite maximiser son score à l'instant final alors que son opposant déplace une particule dans le domaine, la condition de Neumann intervenant par une pénalisation adaptée sur le score au voisinage de la frontière. En s'inspirant d'une approche abstraite proposée par Barles et Souganidis, nous prouvons la convergence en établissant des propriétés de monotonie, stabilité et consistance.

Le second chapitre est consacré à des résultats de régularité sur les solutions d'équations paraboliques non-linéaires associés à un opérateur uniformément elliptique. Tout d'abord, nous donnons une estimation de la mesure de Lebesgue de l'ensemble des points possédant un développement de Taylor quadratique global avec un contrôle sur la taille du terme cubique en espace et en temps. Sous une hypothèse supplémentaire sur la régularité de la non-linéarité, nous en déduisons un résultat de régularité partielle portant sur le caractère  $C^{2,\alpha}$  des solutions.

Dans les troisième et quatrième chapitres, nous proposons une méthode générale pour obtenir des taux algébriques de convergence de solutions de schémas d'approximation vers la solution de viscosité sous l'hypothèse d'uniforme ellipticité de l'opérateur. Dans un premier temps, nous donnons le taux de convergence pour des schémas elliptiques obtenus par principe de programmation dynamique et proposés par Kohn et Serfaty. Nous prouvons ensuite un taux de convergence pour des schémas par différences finies implicites en temps associés à des équations paraboliques.

**Mots-clés :** Solutions de viscosité, équations elliptiques non-linéaires, équations paraboliques non-linéaires, régularité, approximation, problème de Neumann, contrôle déterministe, principe de programmation dynamique, inégalité d'Alexandrov-Bakelman-Pucci, régularité partielle, schémas paraboliques, enveloppe monotone, taux de convergence.

# Abstract

In this thesis we study some approximation and regularity results for viscosity solutions of fully nonlinear elliptic and parabolic equations.

In the first chapter, we consider a broad class of fully nonlinear elliptic and parabolic equations with inhomogeneous Neumann boundary conditions. We provide a deterministic control interpretation through two-person repeated games which represents the solution as the limit of the sequence of the scores associated to the games. The first player wants to maximize his score at the final time while his opponent moves a particle in the domain, the Neumann condition intervening by a suitable penalization on the score near the boundary. Inspired by an abstract method of Barles and Souganidis, we prove the convergence of the score to the solution of the equation by establishing monotonicity, stability and consistency.

The second chapter presents some regularity results about viscosity solutions of parabolic equations associated to a uniformly elliptic operator. First we obtain a Lebesgue measure estimate on the points having a quadratic Taylor expansion with a controlled cubic term in space and time. Under an additional assumption on the regularity of the nonlinearity, we deduce a partial regularity result about the  $C^{2,\alpha}$  regularity of these solutions.

In the third and fourth chapters, we propose a general approach to determine algebraic rates of convergence of solutions of approximation schemes to the viscosity solution of fully nonlinear elliptic or parabolic equations under the assumption of uniform ellipticity of the operator. We first give the rate associated to the elliptic schemes derived by dynamic programming principles and proposed by Kohn and Serfaty. We then prove a rate of convergence for finite-difference schemes implicit in time associated to fully nonlinear parabolic equations.

**Keywords :** Viscosity solutions, fully nonlinear elliptic equations, fully nonlinear parabolic equations, regularity, approximation, Neumann problem, deterministic control, dynamic programming principle, Alexandrov-Bakelman-Pucci Inequality, partial regularity, parabolic schemes, monotone envelope, rates of convergence.